CONCERNING SEMICONNECTED MAPS

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Introduction. Professor John Jones, Jr., [3], introduces a semiconnected map $f: X \to Y$ as one in which f^{-1} preserves closed connected subsets of Y, and gives conditions under which a semiconnected map is continuous or is a homeomorphism. Theorem 1 of that paper is generalized here, and comparisons are made between semiconnected maps and other noncontinuous maps.

Among the several other well-known types of noncontinuous maps only the connected map and the connectivity map will be considered. A connected map $f: X \to Y$ is one which preserves connected subsets of X and a connectivity map $f: X \to Y$ is one for which the induced graph map, $g: X \to X \times Y$ defined by $g(x) = (x, f(x))$ for each $x \in X$, is connected. It is easy to see that if $f: X \to Y$ is continuous, then f is a connectivity map, and if a connectivity map, then also connected. Examples showing the reverse implications are not always valid may be found in [2]. The example $f(x) = x^2$ from the reals into the reals (usual topology in both cases) shows that continuous maps, hence connected and connectivity maps, need not be semiconnected. Furthermore, $f(x) = x$ from the reals (usual topology) to the reals (discrete topology) is semiconnected but not connected, hence not a connectivity nor a continuous map.

Throughout, $\mathrm{cl}(A)$ denotes the closure of the set A.

Results. Theorem 1 generalizes Theorem 1 of [3].

THEOREM 1. *If $f: X \to Y$ is semiconnected and onto the semi-locally-connected space Y, then f is continuous.*

PROOF. Let $B \subset Y$ be open. It will be shown that $f^{-1}(B)$ is open in X. For each point $b \in B$ there exists an open set $V_b \subset B$ such that $Y - V_b$ consists of a finite number of components $C_1^b, C_2^b, \cdots, C_n^b$. Each C_j^b is closed and connected; hence $f^{-1}(C_j^b)$ is closed and connected since f is semiconnected. Thus $\bigcup_{j=1}^n f^{-1}(C_j^b)$ is closed and contains no point of $f^{-1}(V_b)$ so that $X - \bigcup_{j=1}^n f^{-1}(C_j^b) = R_b$ is an open set in X having the property that $f(R_b) = V_b$. Consequently $\bigcup_{b \in B} R_b$ is open in X and furthermore $f^{-1}(B) = \bigcup_{b \in B} R_b$.

THEOREM 2. *Let $f: X \to Y$ be a closed map where $f^{-1}(y)$ is connected for each $y \in Y$. Then if $M \subset Y$ is connected, $f^{-1}(M)$ is connected.*

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PROOF. Considering $M \subset Y$ nondegenerate, suppose $f^{-1}(M) = H \cup K$, separated. Then $f(H) \cup f(K) = M$ and one of the sets, say $f(H)$, has a limit point y_0 of the other, $f(K)$ in this case. Since $f^{-1}(y)$ is connected for each $y \in Y$, $f^{-1}(y_0) \subset H$ and furthermore $f(H) \cap f(K) = \emptyset$. Consequently, because $\mathrm{cl}(K) \cap H = \emptyset$, $y_0 \notin f(\mathrm{cl}(K))$ which contradicts f being closed. The conclusion that $f^{-1}(M)$ is connected follows.

COROLLARY 1. *Let $f: X \to Y$ be a closed semiconnected map where Y is T_1. Then if $M \subset Y$ is any connected set, $f^{-1}(M)$ is connected.*

COROLLARY 2. *Let $f: X \to Y$ be a closed connected map where $f^{-1}(y)$ is connected for each $y \in Y$ and Y is T_1. Then f is semiconnected.*

PROOF. For any closed connected $M \subset Y$, $f^{-1}(M)$ is connected by Theorem 2. By [4] $f^{-1}(M)$ is also closed and hence f is semiconnected.

COROLLARY 3. *If $f: X \to Y$ is a closed continuous map where $f^{-1}(y)$ is connected for each $y \in Y$, and Y is T_1, then f is semiconnected.*

THEOREM 3. *Let $f: X \to Y$ be continuous where $f^{-1}(y)$ is connected for each $y \in Y$, X is countably compact first countable and Y is T_1 first countable. Then f is semiconnected.*

PROOF. Let $M \subset Y$ be closed and connected. Continuity of f insures $f^{-1}(M)$ closed. It will now be shown that $f^{-1}(M)$ is connected from which the conclusion that f is semiconnected follows.

Suppose $f^{-1}(M) = H \cup K$, separated. Then $f(H) \cup f(K) = M$ and one of these sets, say $f(H)$, has a limit point y_0 of the other, $f(K)$ in this instance. There exists a sequence of distinct points $y_n \in f(K)$ such that $y_n \to y_0$ where $f^{-1}(y_0) \subset H$ and $f^{-1}(y_n) \subset K$ for each n. Extracting $x_n \in K \cap f^{-1}(y_n)$ for each n, the set $\{x_n\}$ has a limit point $x_0 \notin H$ since $\mathrm{cl}(K) \cap H = \emptyset$. Thus $f(x_0) \neq y_0$. Since X is first countable, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x_0$. But $f(x_{n_k}) \to y_0 \neq f(x_0)$ contradicting continuity of f [1, Theorem 3.15, p. 102]. Thus $f^{-1}(M)$ is connected.

REFERENCES

1. D. W. Hall and G. L. Spencer II, *Elementary topology*, Wiley, New York, 1957.

2. O. H. Hamilton, *Fixed points for certain non-continuous transformations*, Proc. Amer. Math. Soc. **8** (1957), 750–756.

3. John Jones, Jr., *On semiconnected mappings of topological spaces*, Proc. Amer. Math. Soc. **19** (1968), 174–175.

4. D. E. Sanderson, *Noncontinuous functions which act like continuous functions for connected sets*, Notices Amer. Math. Soc. **3** (1964), 765.

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