CONCERNING SEMICONNECTED MAPS

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Introduction. Professor John Jones, Jr., [3], introduces a semiconnected map \( f: X \to Y \) as one in which \( f^{-1} \) preserves closed connected subsets of \( Y \), and gives conditions under which a semiconnected map is continuous or is a homeomorphism. Theorem 1 of that paper is generalized here, and comparisons are made between semiconnected maps and other noncontinuous maps.

Among the several other well-known types of noncontinuous maps only the connected map and the connectivity map will be considered. A connected map \( f: X \to Y \) is one which preserves connected subsets of \( X \) and a connectivity map \( f: A \to F \) is one for which the induced graph map, \( g: X \to X \times Y \) defined by \( g(x) = (x, f(x)) \) for each \( x \in X \), is connected. It is easy to see that if \( f: X \to Y \) is continuous, then \( f \) is a connectivity map, and if a connectivity map, then also connected. Examples showing the reverse implications are not always valid may be found in [2]. The example \( f(x) = x^2 \) from the reals into the reals (usual topology in both cases) shows that continuous maps, hence connected and connectivity maps, need not be semiconnected. Furthermore, \( f(x) = x \) from the reals (usual topology) to the reals (discrete topology) is semiconnected but not connected, hence not a connectivity nor a continuous map.

Throughout, \( \text{cl}(A) \) denotes the closure of the set \( A \).

Results. Theorem 1 generalizes Theorem 1 of [3].

Theorem 1. If \( f: X \to Y \) is semiconnected and onto the semi-locally-connected space \( Y \), then \( f \) is continuous.

Proof. Let \( P \subseteq F \) be open. It will be shown that \( f^{-1}(P) \) is open in \( X \). For each point \( b \in B \) there exists an open set \( V_b \subseteq B \) such that \( Y - V_b \) consists of a finite number of components \( C_1, C_2, \ldots, C_k \). Each \( C_i \) is closed and connected; hence \( f^{-1}(C_i) \) is closed and connected since \( f \) is semiconnected. Thus \( \bigcup_{i=1}^{k} f^{-1}(C_i) \) is closed and contains no point of \( f^{-1}(V_b) \) so that \( X - \bigcup_{i=1}^{k} f^{-1}(C_i) = R_b \) is an open set in \( X \) having the property that \( f(R_b) = V_b \). Consequently \( \bigcup_{b \in B} R_b \) is open in \( X \) and furthermore \( f^{-1}(B) = \bigcup_{b \in B} R_b \).

Theorem 2. Let \( f: X \to Y \) be a closed map where \( f^{-1}(y) \) is connected for each \( y \in Y \). Then if \( M \subseteq Y \) is connected, \( f^{-1}(M) \) is connected.

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117
Proof. Considering $M \subseteq Y$ nondegenerate, suppose $f^{-1}(M)$
$= H \cup K$, separated. Then $f(H) \cup f(K) = M$ and one of the sets, say
$f(H)$, has a limit point $y_0$ of the other, $f(K)$ in this case. Since $f^{-1}(y)$
is connected for each $y \in Y$, $f^{-1}(y_0) \subseteq H$ and furthermore $f(H) \cap f(K)
= \emptyset$. Consequently, because $\text{cl}(K) \cap H = \emptyset$, $y_0 \notin f(\text{cl}(K))$ which con-
tradicts $f$ being closed. The conclusion that $f^{-1}(M)$ is connected
follows.

Corollary 1. Let $f: X \to Y$ be a closed semiconnected map where $Y$
is $T_1$. Then if $M \subseteq Y$ is any connected set, $f^{-1}(M)$ is connected.

Corollary 2. Let $f: X \to Y$ be a closed connected map where $f^{-1}(y)$
is connected for each $y \in Y$ and $Y$ is $T_1$. Then $f$ is semiconnected.

Proof. For any closed connected $M \subseteq Y$, $f^{-1}(M)$ is connected by
Theorem 2. By [4] $f^{-1}(M)$ is also closed and hence $f$ is semiconnected.

Corollary 3. If $f: X \to Y$ is a closed continuous map where $f^{-1}(y)$ is
connected for each $y \in Y$, and $Y$ is $T_1$, then $f$ is semiconnected.

Theorem 3. Let $f: X \to Y$ be continuous where $f^{-1}(y)$ is connected for
each $y \in Y$, $X$ is countably compact first countable and $Y$ is $T_1$ first
countable. Then $f$ is semiconnected.

Proof. Let $M \subseteq Y$ be closed and connected. Continuity of $f$ insures
$f^{-1}(M)$ closed. It will now be shown that $f^{-1}(M)$ is connected from
which the conclusion that $f$ is semiconnected follows.

Suppose $f^{-1}(M) = H \cup K$, separated. Then $f(H) \cup f(K) = M$ and one of
these sets, say $f(H)$, has a limit point $y_0$ of the other, $f(K)$ in this
instance. There exists a sequence of distinct points $y_n \in f(K)$ such that
$y_n \to y_0$ where $f^{-1}(y_0) \subset H$ and $f^{-1}(y_n) \subset K$ for each $n$. Extracting
$x_n \in K \cap f^{-1}(y_n)$ for each $n$, the set $\{x_n\}$ has a limit point $x_0 \in H$ since
$\text{cl}(K) \cap H = \emptyset$. Thus $f(x_0) \neq y_0$. Since $X$ is first countable, there is a
subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x_0$. But $f(x_{n_k}) \to y_0 \neq f(x_0)$
contradicting continuity of $f$ [1, Theorem 3.15, p. 102]. Thus $f^{-1}(M)$ is
connected.

References

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