ON THE DECOMPOSITION OF INFINITELY DIVISIBLE CHARACTERISTIC FUNCTIONS WITH CONTINUOUS POISSON SPECTRUM

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1. Introduction. Let f be an infinitely divisible characteristic function of the real variable t. It is well known (see, for example, [**3**, Theorem 5.5.2]) that f admits the so-called Lévy's representation

$$\begin{aligned}\log f(t) = ita - \tfrac{1}{2}\sigma^2 t^2 &+ \int_{-\infty}^{0} (e^{itu} - 1 - itu(1+u^2)^{-1})dM(u) \\ &+ \int_{0}^{\infty} (e^{itu} - 1 - itu(1+u^2)^{-1})dN(u),\end{aligned} \tag{1}$$

where M, N, a and σ satisfy the following conditions:

(a) a is a real constant;

(b) σ^2 is a nonnegative constant;

(c) M and N are nondecreasing functions in the intervals $(-\infty, 0)$ and $(0, +\infty)$ respectively;

(d) $M(-\infty) = N(+\infty) = 0$;

(e) $\int_{-\epsilon}^{0} u^2 dM(u) + \int_0^{\epsilon} u^2 dN(u) < +\infty$ for any $\epsilon > 0$.

In the following, we denote the set of all the infinitely divisible characteristic functions without indecomposable factors by I_0 and $S(\mu)$ means the set of the points of increase (or spectrum) of the function μ.

In the case when at least one of the spectra $S(M)$ and $S(N)$ of the functions M and N in the representation (1) contains an interval, the following results are known.

THEOREM A (CRAMÉR [**1**]). *If there exist two positive constants k and c such that at least one of the relations $M'(u) > k$ almost everywhere in $(-c, 0)$ or $N'(u) > k$ almost everywhere in $(0, c)$ holds, then the function f defined by* (1) *does not belong to I_0.*

THEOREM B (SHIMIZU [**5**]). *If there exist three positive constants k, c and d $(d > 2c)$ such that at least one of the relations $M'(u) > k$ almost everywhere in $(-d, -c)$ or $N'(u) > k$ almost everywhere in (c, d) holds, then the function f defined by* (1) *does not belong to I_0.*

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THEOREM C (OSTROVSKIĬ [4]). *If the following conditions are satisfied*:

(a) $\sigma=0$;

(b) $S(M)=\varnothing$;

(c) *there exists a positive constant* c *such that* $S(N)\subset[c, 2c]$, *then the function* f *defined by* (1) *belongs to* I_0.

In the present note, we prove, using a modification of the method of Cramér [1], a result which supplements the preceding theorems and we give some remarks on the structure of I_0.

2. The main result.

THEOREM 1. *Let* f *be the characteristic function defined by*

$$\log f(t) = \int_{-\infty}^{+\infty} (e^{itu} - 1 - itu(1+u^2)^{-1})\alpha(u)du,$$

where

$$\begin{aligned} \alpha(u) &= k \qquad \text{if } -c(1+2^{-n}) < u < -c \quad \text{or} \quad d < u < d(1+2^{-n}), \\ &= 0 \qquad \text{otherwise}, \end{aligned}$$

k, c *and* d *being positive constants and* n *being a positive integer. Then* f *has an indecomposable factor.*

PROOF. We give the demonstration in the particular case $c=d=1$. The proof in the general case follows the same lines, but is more intricate.

Let β be the function

$$\begin{aligned} \beta(u) &= k \qquad \text{if } 1 < |u| < 1+2^{-n}, \\ &= -k\epsilon \qquad \text{if } |u| < 2^{-n}, \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

If we prove that the function g defined by

$$\log g(t) = \int_{-\infty}^{+\infty} (e^{itu} - 1 - itu(1+u^2)^{-1})\beta(u)du \tag{2}$$

is a characteristic function for some $\epsilon>0$, then clearly g divides f and, since g is not infinitely divisible, the theorem will be proved (cf. [3, Theorem 6.2.2]).

Let α_m and β_m be the functions defined by

$$\alpha_1(x) = \alpha(x); \qquad \alpha_m(x) = \int_{-\infty}^{+\infty} \alpha_{m-1}(x-t)\alpha_1(t)dt;$$

$$\beta_1(x) = \beta(x); \qquad \beta_m(x) = \int_{-\infty}^{+\infty} \beta_{m-1}(x-t)\beta_1(t)dt.$$

We denote by $K(m)$ the constant $K(m)=m(1+2^{-n})$ and prove by induction that

$$\alpha_m(x) = \beta_m(x) \qquad \text{if } |x| > K(m) - 1. \tag{3}$$

This is true for $m=1$. Suppose that

$$\alpha_{m-1}(x) = \beta_{m-1}(x) \qquad \text{if } |x| > K(m-1) - 1.$$

From the properties of convolutions, we have

$$\alpha_m(x) = \beta_m(x) = 0 \qquad \text{if } |x| \geqq K(m).$$

Moreover, if $K(m)-1<x<K(m)$, then

$$\alpha_m(x) = \int_{-\infty}^{+\infty} \alpha_{m-1}(x-t)\alpha_1(t)dt = k\int_{-K(1)}^{\rho} \alpha_{m-1}(x-t)dt$$
$$= k\int_{-K(1)}^{\rho} \beta_{m-1}(x-t)dt = \beta_m(x)$$

where $\rho=\inf(-1, K(m-1)-x)$. The case $-K(m)<x<-K(m)+1$ being exactly the same, (3) is proved. Therefore, for any m,

$$\beta_m(x) \geqq 0 \qquad \text{if } |x| \geqq K(m) - 1. \tag{4}$$

Now, we prove that

$$\lim_{\epsilon\to 0} \sup_{|x|\leqq K(m)-1} |\alpha_m(x) - \beta_m(x)| = 0. \tag{5}$$

In fact, if $\epsilon<1$, we have

$$|\alpha_m(x)| < [K(2)]^{m-1}k^m, \qquad |\beta_m(x)| < [K(2)]^{m-1}k^m,$$

and from these formulae and from

$$\alpha_m(x) - \beta_m(x) = \int_{-\infty}^{+\infty} \alpha_{m-1}(x-t)\alpha_1(t)dt - \int_{-\infty}^{+\infty} \beta_{m-1}(x-t)\beta_1(t)dt$$
$$= \int_{-\infty}^{+\infty} \alpha_{m-1}(x-t)[\alpha_1(t) - \beta_1(t)]dt$$
$$- \int_{-\infty}^{+\infty} [\beta_{m-1}(x-t) - \alpha_{m-1}(x-t)]\beta_1(t)dt,$$

it follows by induction that

$$|\alpha_m(x) - \beta_m(x)| < \epsilon(2k)^m[K(2)]^{m-1}$$

and (5) is proved.

The next step is the proof of the formula

$$S(\alpha_m) = \bigcup_{j=0}^{m} [j - K(m-j), K(j) + j - m]. \tag{6}$$

This is true for $m=1$. Now $S(\alpha_{m+1}) = S(\alpha_m)(+)A$, where

$$A = [-K(1), -1] \cup [1, K(1)],$$

and $(+)$ indicates the vectorial sum. Therefore

$$\begin{aligned} S(\alpha_{m+1}) &= \bigcup_{j=0}^{m} [j - K(m-j) - K(1), K(j) + j - m - 1] \\ &\quad \cup \bigcup_{j=0}^{m} [j - K(m-j) + 1, K(j) + j - m + K(1)] \\ &= \bigcup_{j=0}^{m} [j - K(m+1-j), K(j) + j - m - 1] \\ &\quad \cup \bigcup_{j=1}^{m+1} [j - K(m+1-j), K(j) + j - m - 1] \end{aligned}$$

and (6) is proved.

(6) implies that $S(\alpha_m)$ is all the interval $[-K(m), K(m)]$ if $m \geqq c(1)$ where $c(j)$ means the constant $2^{n+1}+j$. But, from the definition of α_m as convolution of nonnegative functions, we have

$$\inf_{|x| < K(c(j))-1} \alpha_{c(j)}(x) > 0 \qquad j = 2, \cdots, c(3)$$

and

$$\inf_{|x| < K(c(2))-1} \left[\sum_{j=1}^{c(2)} \frac{1}{j!} \alpha_j(x) \right] > 0.$$

Therefore, from (5), if ϵ is small enough,

$$\inf_{|x| < K(c(j))-1} \beta_{c(j)}(x) > 0, \qquad j = 2, \cdots, c(3) \tag{7}$$

and

$$\inf_{|x| < K(c(2))-1} \left[\sum_{j=1}^{c(2)} \frac{1}{j!} \beta_j(x) \right] > 0. \tag{8}$$

From the definition of β_m, we have for $k < m$

$$\beta_m(x) = \int_{-\infty}^{\infty} \beta_{m-k}(x - t)\beta_k(t)dt$$

and from (4), (7) and the properties of convolutions it follows that if ϵ is small enough

$$\beta_m(x) \geqq 0, \qquad m > c(2)$$

and this, with (8), implies that

$$\sum_{j=1}^{\infty} \frac{1}{j!} \beta_j(x) > 0 \tag{9}$$

for any x if ϵ is small enough. Letting

$$G(x) = e^{-\lambda} \left\{ \chi(x + \eta) + \int_{-\infty}^{x} \left[\sum_{j=1}^{\infty} \frac{1}{j!} \beta_j(y + \eta) \right] dy \right\}$$

where $\chi(x)$ is the degenerate distribution and

$$\lambda = \int_{-\infty}^{+\infty} \beta(x)dx,$$

$$\eta = \int_{-\infty}^{+\infty} x(1 + x^2)^{-1}\beta(x)dx,$$

then G is, from (9), a distribution function if ϵ is small enough and it is easily shown that the belonging characteristic function is g. That completes the proof of the theorem.

From this theorem and from Theorems A and B of the introduction, we deduce easily the

COROLLARY. *Let f be an infinitely divisible characteristic function and suppose that the functions M and N which occur in its Lévy's representation satisfy the following condition: There exist five constants a, b, c, d and k $(0 \leqq a < b, 0 \leqq c < d, k > 0)$ such that $M'(u) > k$ almost everywhere in $(-b, -a)$ and $N'(u) > k$ almost everywhere in (c, d). Then f has an indecomposable factor.*

3. Remarks. Let f be the infinitely divisible characteristic function defined by (1). If we define the functions f_N, f_+ and f_- by

$$\log f_N(t) = ita - \tfrac{1}{2}\sigma^2 t^2,$$

$$\log f_+(t) = \int_0^{+\infty} (e^{itu} - 1 - itu(1 + u^2)^{-1})dN(u),$$

$$\log f_-(t) = \int_{-\infty}^{0} (e^{itu} - 1 - itu(1 + u^2)^{-1})dM(u),$$

we have a natural decomposition of f, $f=f_N f_+ f_-$, as product of three infinitely divisible characteristic functions.

It is well known that we can have the following circumstance: f_+ belongs to I_0, $f_-=1$ while f does not belong to I_0 (clearly f_N belongs to I_0). Such examples can be constructed, for example, from the Theorem 8.1.1 of Linnik [2].

From the particular case of the preceding corollary when $b \leqq 2a$, $d \leqq 2c$ and from the theorem of Ostrovskiĭ stated in the introduction, we have examples of another circumstance: f_+ and f_- belong to I_0 while, for $f_N=1$, f does not belong to I_0.

Such examples with discontinuous Poisson spectra are given by the

THEOREM 2. *Let f be the infinitely divisible characteristic function defined by*

$$\log f(t) = \lambda[\exp(ipt) + \exp(-ipt) - 2] + \mu[\exp(i(p+2)t) + \exp(-i(p+2)t) - 2]$$

where p is an integer greater than two and λ and μ are positive constants. Then f has an indecomposable factor.

PROOF. The method is a modification of one of P. Lévy (see [3, pp. 178–179]). We distinguish two cases according as p is even or odd.

(a) $p=2q$. Let P and Q be the functions defined by

$$P(x) = 1 + \lambda(x^q + x^{-q}) + \mu(x^{q+1} + x^{-q-1}), \quad Q(x) = P(x) - \epsilon(x + x^{-1}),$$

where ϵ is positive. Then

$$Q^m(x) = P^m(x) + \sum_{k=0}^{(m-1)(q+1)+1} \eta_{m,k}(\epsilon)(x^k + x^{-k}) \tag{10}$$

where

$$\lim_{\epsilon \to 0} \eta_{m,k}(\epsilon) = 0.$$

But it is easily shown that every integer k satisfying $|k| \leqq q(q+1)$ can be written in the form $k=rq+s(q+1)$, where r and s are integers satisfying

$$|r| + |s| \leqq q.$$

Therefore

$$P^{q+j}(x) = \sum_{k=0}^{(q+j)(q+1)} \alpha_{q+j,k}(x^k + x^{-k}), \qquad j = 0, 1, \cdots, q-1, \tag{11}$$

and

$$\sum_{j=0}^{q} \frac{1}{j!} P^j(x) = \sum_{k=0}^{q(q+1)} \beta_k(x^k + x^{-k}), \tag{12}$$

where $\alpha_{q+j,k} > 0$, $\beta_k > 0$. But (10), (11), (12) imply

$$\exp[Q(x)] = \sum_{k=0}^{\infty} \gamma_k(x^k + x^{-k}) \tag{13}$$

where $\gamma_k \geqq 0$ if ϵ is small enough. It follows that the function g defined by

$$g(t) = \exp[Q(e^{2it}) - Q(1)]$$

is a characteristic function. Clearly g divides f and g is not infinitely divisible. The theorem is proved in this case.

(b) $p = 2q - 1$. We define now P and Q by

$$P(x) = 1 + \lambda(x^{2q-1} + x^{1-2q}) + \mu(x^{2q+1} + x^{-2q-1}),$$
$$Q(x) = P(x) - \epsilon(x + x^{-1}),$$

where ϵ is positive. Then, using the fact that every integer k satisfying $|k| \leqq 4q^2$ can be written in the form $k = r(2q-1) + s(2q+1)$ where r and s are integers satisfying

$$|r| + |s| \leqq 2q,$$

we obtain evident analogues of (10), (11) and (12) and therefore the validity of (13) if ϵ is small enough. It follows that the function g defined by

$$g(t) = \exp[Q(e^{it}) - Q(1)]$$

is a characteristic function and, since g divides f and is not infinitely divisible, the theorem is proved.

With the same method, we can prove that the following infinitely divisible characteristic functions do not belong to I_0:

(a) $$f_1(t) = \exp[\lambda_1(e^{ipt} - 1) + \lambda_2(e^{i(p+1)t} - 1) + \lambda_3(e^{-iqt} - 1) + \lambda_4(e^{-i(q+1)t} - 1)],$$

where p and q are integers greater than one and λ_j are positive constants ($j = 1, 2, 3, 4$);

(b) $$f_2(t) = \exp[\mu_1(e^{ipt} - 1) + \mu_2(e^{i(p+1)t} - 1) + \mu_3(e^{-it} - 1)],$$

where p is an integer greater than one and μ_j are positive constants ($j = 1, 2, 3$);

(c) $$f_3(t) = \exp[\nu_1(e^{ipt} - 1) + \nu_2(e^{i(p+1)t} - 1) + \nu_3(e^{it} - 1)],$$

where p is an integer greater than two and ν_j are positive constants ($j=1, 2, 3$).

Finally, we mention an open problem: Let f be an infinitely divisible characteristic function such that $f_N f_+$ and $f_N f_-$ belong to I_0. If f_N is nondegenerate, is it possible that f does not belong to I_0?

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