GENERALIZATION OF A THEOREM OF PÓSA

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A *hamiltonian cycle* in a graph G is a cycle containing all the points of G, and a graph with a hamiltonian cycle is called *hamiltonian*. In [**3**], Pósa proved the following interesting and important theorem.

THEOREM OF PÓSA. *Let G be a graph on p $(\geqq 3)$ points such that for every integer i with $1 \leqq i < p/2$, the number of points of degree not exceeding i is less than i. Then G is hamiltonian.*

A graph G on p $(\geqq 3)$ points is said to be *k-path hamiltonian* if every path of length not exceeding k, $0 \leqq k \leqq p-2$, is contained in a hamiltonian cycle of G. The o-path hamiltonian graphs are then the hamiltonian graphs. The object of this note is to generalize Pósa's theorem to k-path hamiltonian graphs. The proof is an extension of that used in the proof of Theorem 1 of [**2**].

THEOREM. *Let G be a graph with p $(\geqq 3)$ points, and let $0 \leqq k \leqq p-3$. If for every integer i with $k+1 \leqq i < (p+k)/2$, the number of points of degree not exceeding i is less than $i-k$, then G is k-path hamiltonian.*

PROOF. Assume that G satisfies the hypothesis of the theorem but contains a path P of length not exceeding k which is not contained in a hamiltonian cycle. We may assume that G becomes k-path hamiltonian whenever any new line is added to G. For if G did not originally have this property we could add suitable lines until it did and the resulting graph would still satisfy the hypothesis of the theorem.

Let v_1 and v_p be two nonadjacent points of G such that (1) $\rho(v_1) \leqq \rho(v_p)$, where $\rho(v)$ denotes the degree of the point v, and (2) $\rho(v_1) + \rho(v_p)$ is as large as possible. If we add the line v_1v_p to G we obtain a k-path hamiltonian graph G'. Let C be a hamiltonian cycle of G' which contains the path P. Clearly C must include the line v_1v_p and hence v_1 and v_p are the endpoints of a spanning path $Q = (v_1, v_2, \cdots, v_p)$ in G which contains the path P. If v_i, $2 \leqq i < p$, is adjacent to v_1 and if $v_{i-1}v_i$ is not in P, then $v_{i-1}v_p$ is not in G. For otherwise, $(v_1, v_i, v_{i+1}, \cdots, v_p, v_{i-1}, v_{i-2}, \cdots, v_1)$ would be a hamiltonian cycle of G containing P. Since at most k lines of Q belong to P, it follows that there are at least $\rho(v_1) - k$ points in G which are nonadjacent to v_p. Therefore, $\rho(v_1) \leqq \rho(v_p) \leqq (p-1) - (\rho(v_1) - k)$ so that $\rho(v_1) \leqq (p+k-1)/2$. Furthermore, whenever v_i is adjacent to v_1 and $v_{i-1}v_i$

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is not in P, $(v_{i-1}, v_{i-2}, \cdots, v_1, v_i, v_{i+1}, \cdots, v_p)$ is a spanning path in G containing P. By the manner in which v_1 and v_p were chosen, it follows that $\rho(v_{i-1}) \leqq \rho(v_1)$. Thus, there are at least $\rho(v_1)-k$ points having degree not exceeding $\rho(v_1)$. However, $k+1 \leqq \rho(v_1) \leqq (p+k-1)/2 < (p+k)/2$ so that by assumption there are less that $\rho(v_1)-k$ points having degree not exceeding $\rho(v_1)$. Having been led to a contradiction, we conclude that the theorem is true.

COROLLARY. *If G is a graph with p $(\geqq 3)$ points such that each point has degree at least $(p+k)/2$, $0 \leqq k \leqq p-2$, then G is k-path hamiltonian.*

It is not difficult to construct examples that show that the theorem and its corollary are each, in a sense, best possible. However, the problem of finding conditions which are both necessary and sufficient for a graph to be k-path hamiltonian remains unsolved and appears to be extremely difficult. It was, however, shown in [1] that a graph G is $(p-2)$-path hamiltonian if and only if G is (1) the cycle C_p, (2) the complete graph K_p, or (3) the complete bipartite graph $K(p/2, p/2)$, where (3) is possible only if p is even.

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