INDUCED REPRESENTATIONS OF LIE ALGEBRAS. II

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1. Introduction. Let \( \mathfrak{g} \) be a Lie algebra over a field \( K \). A decomposition of \( \mathfrak{g} \) is a triple \( (n_1, \mathfrak{h}, n_2) \) of subalgebras of \( \mathfrak{g} \) such that \( \mathfrak{g} = n_1 \oplus \mathfrak{h} \oplus n_2 \) a vector space direct sum and such that \( [\mathfrak{h}, n_i] \subseteq n_i \) for \( i = 1, 2 \). In [5] we showed how one could "induce" \( \mathfrak{g} \)-modules from \( \mathfrak{h} \)-modules in a natural manner. In this paper we prove a useable necessary and sufficient condition that a \( \mathfrak{g} \)-module must satisfy in order that it be "induced" from an \( \mathfrak{h} \)-module. We apply this method of induction to obtain all well-known simple modules (not necessarily finite dimensional) for semisimple Lie algebras over algebraically closed fields of characteristic 0.

2. Preliminary results. Let \( \mathfrak{g} \) be a Lie algebra over a field \( K \) with decomposition \( (n_1, \mathfrak{h}, n_2) \). Set \( \mathfrak{f} = n_1 + \mathfrak{h} \). Let \( U(\mathfrak{g}) \) be the universal enveloping algebra of \( \mathfrak{g} \) and let \( U(\mathfrak{f}), U(n_1), U(\mathfrak{h}) \) and \( U(n_2) \) be the universal enveloping algebras of \( \mathfrak{f}, n_1, \mathfrak{h}, n_2 \) canonically embedded in \( U(\mathfrak{g}) \). We look upon \( U(\mathfrak{g}) \) as a left \( U(\mathfrak{f}) \)-module and a right \( U(\mathfrak{g}) \)-module under, respectively, left and right multiplication. Let \( W \) be an \( \mathfrak{h} \)-module. We denote by \( \hat{W} \) the \( \mathfrak{f} \)-module with space \( W \) and with \( n_1 \cdot W = 0 \). We set \( T(W) = \text{Hom}_{U(\mathfrak{f})}(U(\mathfrak{g}), \hat{W}) \) with the natural left \( \mathfrak{g} \)-module structure:

\[
(x \cdot f)(g) = f(gx) \quad \text{for} \quad x \in \mathfrak{g}, \quad f \in T(W), \quad g \in U(\mathfrak{g}).
\]

By the Poincaré-Birkhoff-Witt (abbreviated P-B-W) theorem \( U(\mathfrak{g}) = U(\mathfrak{f})n_1 \cdot U(n_2) \oplus U(\mathfrak{f}) \) a right \( \mathfrak{g} \)-module, left \( \mathfrak{f} \)-module direct sum. Let \( \gamma: U(\mathfrak{g}) \to U(\mathfrak{f}) \) be the corresponding \( \mathfrak{h} \)-module projection. Let \( w: W \to T(W) \) be defined by \( w(v)(g) = \gamma(g) \cdot v \) for \( g \in U(\mathfrak{g}), \ v \in W \). It is easy to see that \( w(W) = T(W) \circ \gamma \) (see [5]) and is thus isomorphic with \( W \) as an \( \mathfrak{h} \)-module. We set \( W^* = U(\mathfrak{g}) \cdot w(W) \). \( W^* \) is a \( \mathfrak{g} \)-submodule of \( T(W) \) and \( W^* \) is the "induced" module of §1. Before proceeding, we introduce one more bit of notation. If \( V \) is a left \( \mathfrak{g} \)-module and if \( M \) is a subalgebra of \( \mathfrak{g} \) then set \( V^M = \{ v \in V \mid M \cdot v = 0 \} \).

**Lemma 2.1.** Let \( W \) be an \( \mathfrak{h} \)-module.

1. \((W^*)^n \neq (0)\) if \( W^* \neq (0) \).
2. \( U(\mathfrak{g}) \cdot (W^*)^n = W^* \).

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(3) \((W^*)^{n_1} \cap n_1 \cdot W^* = (0)\).

(4) \(n_1 \cdot W^*\) contains no nonzero \(\mathfrak{g}\)-modules.

**Proof.** \(w(W) = (W^*)^{n_1}\) and thus (1) is true. Statement (2) is just the definition of \(W^*\). To prove statement (3) we notice that if \(f \in (W^*)^{n_1}\) then \(f \neq 0\) if and only if \(f(1) \neq 0\). If \(n_1 \cdot f \not\in (W^*)^{n_1}\) for some \(f \in W^*\) and \(n_1 \not\in n_1\) then \((n_1 \cdot f)(1) = f(n_1) = n_1 \cdot f(1) = 0\). Thus \(n_1 \cdot f = 0\). We have thus proven (3). If \(f \not\in n_1 \cdot W^*\) and if \(U(\mathfrak{g}) \cdot f \not\subset n_1 \cdot W^*\) then \((g \cdot f) \circ \gamma = 0\) for all \(g \in U(\mathfrak{g})\). In fact, if \((g \cdot f) \circ \gamma \neq 0\) for some \(g \in U(\mathfrak{g})\) then \(g \cdot f \circ \gamma \in (W^*)^{n_1}\) (by the remarks preceding the lemma) and \(g \cdot f \not\in n_1 \cdot W^*\) implies that \(g \cdot (f \circ \gamma) \not\in n_1 \cdot W^* \cap (W^*)^{n_1} = (0)\), which is a contradiction. We thus have \(0 = (g \cdot f)(\gamma(1)) = (g \cdot f)(1) = f(g)\) for all \(g \in U(\mathfrak{g})\). Hence \(f = 0\). The lemma is thus proved.

Lemma 1 implies that as an \(\mathfrak{h}\)-module, \(W^* = (W^*)^{n_1} \oplus n_1 \cdot W^*\). Let \(P: W^* \to (W^*)^{n_1}\) and \(Q: W^* \to n_1 \cdot W^*\) be the corresponding \(\mathfrak{h}\)-module projections.

**Lemma 2.2.** Let \(V\) be a \(\mathfrak{g}\)-module and let \(W\) be an \(\mathfrak{h}\)-module. Then the map \(\text{Hom}_{U(\mathfrak{h})}(V, W^*) \to \text{Hom}_{U(\mathfrak{h})}(V, W)\) given by \(f \to P \circ f\) is 1-1. (Here \(W^{*n_1}\) is identified with \(W\) as an \(\mathfrak{h}\)-module.)

**Proof.** If \(P \circ f \neq 0\) then \(f(V) \subset n_1 \cdot W^*\). Lemma 1 (4) tells us that \(f(V) = 0\). Thus \(f = 0\).

We note that Lemma 2 is formally half of the Frobenius reciprocity theorem. We conclude this section with a useful sufficient condition for simplicity of \(W^*\).

**Proposition 2.1.** Let \(W\) be a simple \(\mathfrak{h}\)-module. Suppose that \(n_1 \cdot W^*\) contains no \(\mathfrak{h}\)-modules isomorphic with \(W\). Then \(W^*\) is simple.

**Proof.** Let \(M\) be a nonzero \(\mathfrak{g}\)-submodule of \(W^*\). Let \(P\) and \(Q\) be as above. By Lemma 2.1 (4), \(P(M) \neq (0)\). Thus \(P(M) = W^{*n_1}\) since \(W^{*n_1}\) is simple. If \(M \neq W^*\) then there is a \(v \in M\) such that \(P(v) \neq 0\) and \(Q(v) \neq 0\). In fact, if there is a \(v \in M\) such that \(Q(v) = 0\), \(P(v) \neq 0\) then \(v \in W^{*n_1}\) and by definition of \(W^*\) and simplicity of \(W\), \(U(\mathfrak{g}) \cdot v = W^*\). Let \(v \in M\) such that \(P(v) \neq 0\) and \(Q(v) \neq 0\). If \(h \cdot P(v) \neq 0\) and \(h \cdot Q(v) = 0\) for some \(h \in U(\mathfrak{h})\) then by the above argument \(M = W^*\). Thus there is an element \(v\) in \(M\) such that \(P(v) \neq 0\) and such that if \(h \cdot P(v) \neq 0\) then \(h \cdot Q(v) \neq 0\). Now let \(\tilde{W} = U(\mathfrak{h}) \cdot Q(v) \subset n_1 \cdot W^*\). We define a map \(\xi: \tilde{W} \to W^{*n_1}\) by setting \(\xi(h \cdot Q(v)) = h \cdot P(v)\) for each \(h \in U(\mathfrak{h})\). We show that \(\xi\) is well defined. If \(h \cdot Q(v) = h' \cdot Q(v)\) then \((h - h')Q(v) = 0\). Thus \((h - h')P(v) = 0\) and thus \(\xi(h \cdot Q(v)) = \xi(h' \cdot Q(v))\). Thus \(\xi\) is well defined and injective. \(\xi\) is now clearly an \(\mathfrak{h}\)-module isomorphism. This contradiction implies that \(M = W^*\).
3. An imprimitivity theorem. In this section we prove the converse of Lemma 2.1. We maintain the notation of §2.

**Theorem 3.1.** If $V$ is a $\Theta$-module such that
1. $V^{\Theta} \neq (0)$ if $V \neq (0)$,
2. $U(\Theta) \cdot V^{\Theta} = V$,
3. $n_1 \cdot V \cap V^{\Theta} = (0)$,
4. $n_1 \cdot V$ contains no nonzero $\Theta$-submodules of $V$ then $V$ is $\Theta$-isomorphic with $(V^{n_2})^*$.

**Proof.** By the P-B-W theorem $U(\Theta) = U(n_2 + \mathfrak{h}) + n_1 \cdot U(n_1 + \mathfrak{h})$ a left $\mathfrak{h}$-module direct sum. Let $\gamma_1: U(\Theta) \to U(n_1 + \mathfrak{h})$ and $\gamma_2: U(\Theta) \to n_1 \cdot U(n_1) \cdot U(n_2 + \mathfrak{h})$ be the corresponding $\mathfrak{h}$-module projections. Suppose that $v \in V$. Then $v = g \cdot \mathfrak{h}$ for some $v \in V^{\Theta}$ by (2). Thus $v = \gamma_1(g) \cdot \mathfrak{h} + \gamma_2(g) \cdot \mathfrak{h}$ with $\gamma_1(g) \cdot \mathfrak{h} \in V^{\Theta}$ and $\gamma_2(g) \cdot \mathfrak{h} \in n_1 \cdot V$. Now (3) implies that $V = V^{\Theta} \oplus n_1 \cdot V$ an $\mathfrak{h}$-module direct sum. Let $R: V \to V^{\Theta}$ be the corresponding $\mathfrak{h}$-module projection. We define $\delta: V \to T(V^{n_2})$ by $\delta(v)(g) = R(g \cdot v)$. Then $\delta(v)(kg) = R(kg \cdot v) = 0$ if $k \in n_1 \cdot U(\Theta)$. $\delta(v)(kg) = R(kg \cdot v) = k \cdot R(gv)$ if $k \in \mathfrak{h}$. Thus $\delta(v) \in T(W)$. Now $\delta(g_0 \cdot v)(g) = R(gg_0 \cdot v) = \delta(v)(gg_0) = (g_0 \cdot \delta(v))(g)$ and thus $\delta: V \to T(V)$ is a $\Theta$-module homomorphism. If $\delta(v) = 0$ then $\delta(v)(g) = 0$ for all $g \in U(\Theta)$ and thus $R(g \cdot v) = 0$ for all $g \in U(\Theta)$. This says that $U(\Theta) \cdot v \subset n_1 \cdot V$ and hence $v = 0$. Thus $\delta$ is injective. Suppose that $v \in V^{\Theta}$ then $\delta(v)(g) = R(g \cdot v) = \gamma(g) \cdot v$ (here $v$ is looked upon as an element of $V^{n_2}$). Thus $\delta |_{V^{\Theta}} = 0$. This clearly implies that

$\delta(V) = U(\Theta) \omega(V^{n_2}) = (V^{n_2})^*$. Q.E.D.

As a corollary to Theorem 3.1 we derive the main result of [5] without using the technique of “double dualization.”

**Corollary 3.1.** Suppose that $V$ is a finite dimensional simple $\Theta$-module and that $n_1$ and $n_2$ act nilpotently on $V$. Then $V$ is isomorphic with $(V^{n_2})^*$ as a $\Theta$-module.

**Proof.** We show that $V$ satisfies conditions (1)–(4) of Theorem 3.1. If $V = (0)$ then $V = (0)^*$. If $V \neq (0)$ then by assumption $V^{n_2} \neq (0)$. Since $V$ is simple, $U(\Theta) \cdot V^{\Theta} = V$. If $n_1 \cdot V \cap V^{\Theta} \neq (0)$ then

$V = U(\Theta) \cdot (n_1 \cdot V \cap V^{\Theta}) = U(n_1) \cdot (n_1 \cdot V \cap V^{\Theta}) \subset U(n_1) n_1 \cdot V$.

But $n_1$ acts nilpotently on $V$; thus the above inclusion implies the contradiction $V = (0)$. Thus $n_1 \cdot V \cap V^{\Theta} = (0)$. Finally, since $n_1 \cdot V$ is a proper subspace of $V$, $n_1 \cdot V$ cannot contain any nonzero $\Theta$-submodule of $V$.

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4. The standard decomposition of a semisimple Lie algebra. In this section we assume that \( \mathfrak{g} \) is semisimple and that \( K \) is of characteristic 0 and algebraically closed. Let \( \mathfrak{h} \) be a Cartan subalgebra of \( \mathfrak{g} \) and let \( \Delta \) be the root system of \( \mathfrak{g} \) with respect to \( \mathfrak{h} \). That is, let \( \Delta \) be the set of all linear forms \( \alpha \) on \( \mathfrak{h} \) such that the set \( \mathfrak{g}_\alpha = \{ x \in \mathfrak{g} \mid [h, x] = \alpha(h) \cdot x \text{ for all } h \in \mathfrak{h} \} \) is nonzero. Let \( \pi \) be a set of linearly independent elements of \( \Delta \) such that every element of \( \Delta \) can be written as an integral combination of the elements of \( \pi \) with the coefficients all having the same sign. Such systems always exist, see e.g. Jacobson [3]. Let \( \prec \) be a linear order on \( \Delta \) corresponding to \( \pi \) (i.e. \( \alpha \prec 0 \) if \( \alpha = \sum \gamma \) with \( \sum \gamma > 0 \)). Let \( n^+ = \sum_{\alpha \succ 0} \mathfrak{g}_\alpha, \ n^- = \sum_{\alpha \lt 0} \mathfrak{g}_\alpha. \) Then \( \mathfrak{g} = n^- \oplus \mathfrak{h} \oplus n^+ \) is a decomposition of \( \mathfrak{g} \).

Proposition 4.1. If \( W \) is a simple finite dimensional (= 1 dimensional) \( \mathfrak{h} \)-module then \( W^* \) is a simple \( \mathfrak{g} \)-module.

Proof. \( W^* = U(\mathfrak{h}) \cdot W^{*n+} = U(n^-) \cdot W^{*n+} \oplus n^- \cdot U(n^-) \cdot W^{*n+} \).

Now consider \( n^- \cdot U(n^-) \) as an \( \mathfrak{h} \)-module under the action \( \text{ad}(h)n = h \cdot n - n \cdot h \) for \( n \in n^- \cdot U(n^-) \).

By the P-B-W theorem \( n^- \cdot U(n^-) \) is completely reducible as an \( \mathfrak{h} \)-module and \( n^- \cdot U(n^-) = \sum V_\gamma \) where \( V_\gamma = \{ n \in n^- \cdot U(n^-) \mid \text{ad}(h)n = \gamma(h) \cdot n, h \in \mathfrak{h} \} \) and the above sum is an \( \mathfrak{h} \)-module direct sum where the \( \gamma \)'s are taken to be all nonnegative integral combinations of positive roots. Furthermore \( \dim_K V_\gamma = \rho(\gamma) \). \( \rho(\gamma) \) is the number of ways that \( \gamma \) can be written as a sum \( \gamma = \gamma_1 + \cdots + \gamma_\tau \) where \( \gamma_i \in \Delta, 0 < \gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_\tau \). Thus \( n^- \cdot U(n^-) \) is isomorphic with \( \sum_\gamma \rho(\gamma) K_\gamma \otimes W \) where \( K_\gamma \) is the \( \mathfrak{h} \)-module \( K \) with action \( h \cdot 1 = \gamma(h) \). \( \rho(\gamma) \) is the corresponding multiplicity. \( K_\gamma \otimes W \) is a simple \( \mathfrak{h} \)-module.

Let \( \phi : n^- \cdot U(n^-) \otimes W^{*n+} \rightarrow n^- \cdot U(n^-) \cdot W^{*n+} \) then \( \phi \) is an \( \mathfrak{h} \)-module homomorphism. Thus \( n^- \cdot W^* = n^- \cdot U(n^-) \cdot W^{*n+} \) is \( \mathfrak{h} \)-isomorphic with \( \sum_\gamma M_\gamma K_\gamma \otimes W \) where the \( M_\gamma \) correspond as above to multiplicities. Let \( P_\gamma : n^- \cdot W^* \rightarrow M_\gamma K_\gamma \otimes W \) be the corresponding \( \mathfrak{h} \)-module projection. Suppose that \( W^* \) is an \( \mathfrak{h} \)-submodule of \( n^- \cdot W \) such that \( W^* \) is \( \mathfrak{h} \)-isomorphic with \( W \). Let \( \gamma \) be such that \( P_\gamma(W^*) \neq 0 \). Since \( W^* \) is simple \( P_\gamma |_{W^*} \) is an \( \mathfrak{h} \)-module injection. Thus \( W^* \) is \( \mathfrak{h} \)-isomorphic with \( K_\gamma \otimes W \), for some \( \gamma \neq 0 \), which is impossible. Thus \( W^* \) is simple by Theorem 3.1.

If \( W \) is the \( \mathfrak{h} \)-module \( K_\lambda \) where \( \lambda \in \mathfrak{h}^* \), the dual space of \( \mathfrak{h} \), and \( K_\lambda \) is the \( \mathfrak{h} \)-module \( K \) with the action \( h \cdot 1 = \lambda(h) \), then we set \( W^* = V^\lambda \). We notice that \( (V^\lambda)^{*n+} = \{ v \in V^\lambda \mid h \cdot v = \lambda(h)v, h \in \mathfrak{h} \} = \omega(K_\lambda) \) and thus \( \dim (V^\lambda)^{*n+} = 1 \). Furthermore \( V^\lambda = U(\mathfrak{h}) \cdot (V^\lambda)^{*n+} = U(n^-) \cdot (V^\lambda)^{*n+} \). Thus by the techniques of the proof of Proposition 4.1 we find that \( V^\lambda \)
5. A multiplication on $U(\mathfrak{g})^0$. Let $\mathfrak{g}$ be a Lie algebra over a field $K$. Let $U(\mathfrak{g})$ be its universal enveloping algebra and let $U(\mathfrak{g})^0$ be the dual space of $U(\mathfrak{g}) (= \text{Hom}_K(U(\mathfrak{g}), K))$. Let $\epsilon: U(\mathfrak{g})^0 \otimes U(\mathfrak{g})^0 \rightarrow K$ be given by $\epsilon(f \otimes g) = f(1)g(1)$ and extended by linearity. $U(\mathfrak{g})^0$ is a left $\mathfrak{g}$-module under the action $(x \cdot f)(g) = f(gx)$ where $f \in U(\mathfrak{g})^0$, $x \in \mathfrak{g}$, $g \in U(\mathfrak{g})$. We look upon $U(\mathfrak{g})^0 \otimes U(\mathfrak{g})^0$ as the tensor product $\mathfrak{g}$-module (hence $U(\mathfrak{g})$-module). We can now define a multiplication on $U(\mathfrak{g})^0$: let $f, f' \in U(\mathfrak{g})^0$ then $f \cdot f'(g) = \epsilon(g \cdot (f \otimes f'))$. It is not hard to check that $U(\mathfrak{g})^0$ is an associative, commutative ring under this multiplication. $U(\mathfrak{g})^0$ contains a unit. In fact, $U(\mathfrak{g}) = K \cdot 1 \otimes \mathfrak{g} \cdot U(\mathfrak{g})$ let $f_0: U(\mathfrak{g}) \rightarrow K$ be the corresponding projection. Then $f_0 \cdot f = f$ for all $f \in U(\mathfrak{g})^0$. If $K$ is of characteristic 0 then $U(\mathfrak{g})^0$ is an integral domain. (If $K$ is of characteristic $p \neq 0$ then $U(\mathfrak{g})^0$ has no elements $f$ such that $f^p = f$.) For details see Cartier [1] or Hochschild [2]. We also note that the map $U(\mathfrak{g})^0 \otimes U(\mathfrak{g})^0 \rightarrow U(\mathfrak{g})^0$ given by $f \otimes f' \mapsto f \cdot f'$ is a $\mathfrak{g}$-module homomorphism. Thus $\mathfrak{g}$ acts on $U(\mathfrak{g})^0$ as derivations.

We now return to the case $\mathfrak{g}$ semisimple and $K$ characteristic 0 and algebraically closed, and we maintain the notation of §4. If $V$ is a $\mathfrak{g}$-module and $\mu, \mu' \in \mathfrak{h}^0$ we set (as before) $V_{\mu} = \{v \in V | h \cdot v = \mu(h) v\}$. We also note that if $\lambda \in \mathfrak{h}^0$ then (setting) $\mathcal{T}(\lambda) = T(\lambda) \subset U(\mathfrak{g})^0$. Set $\mathcal{U} U(\mathfrak{g})^0 = \{f \in U(\mathfrak{g})^0 | f(ng) = 0 \text{ for all } n \in \mathbb{N}, g \in U(\mathfrak{g})\}$.

**Lemma 5.1.** (a) $\mathcal{U} U(\mathfrak{g})^0$ is a subring of $U(\mathfrak{g})^0$.
(b) If $\lambda, \lambda', \mu, \mu' \in \mathfrak{h}^0$ then $T(\lambda) \cdot T(\lambda') \subset T(\lambda + \lambda')_{\mu + \mu'}$.

**Proof.** (a) Let $n \in \mathbb{N}, g \in U(\mathfrak{g})$, $f, f' \in \mathcal{U} U(\mathfrak{g})^0$. Then $f \cdot f'(ng) = \epsilon(ng (f \otimes f'))$. Let $g \cdot (f \otimes f') = \sum f_i \otimes f'_i$, $f_i, f'_i \in \mathcal{U} U(\mathfrak{g})^0$ (if $f \in \mathcal{U} U(\mathfrak{g})^0$ then $g \cdot f \in \mathcal{U} U(\mathfrak{g})^0$), $\epsilon(g (f \otimes f')) = \epsilon(n \cdot \sum f_i \otimes f'_i) = \epsilon(\sum (n \cdot f_i \otimes f'_i + f_i \otimes n \cdot f'_i)) = \sum (f_i (n) \cdot f'_i (1) + f_i (1) f'_i (n)) = 0$. Thus $f \cdot f' \in \mathcal{U} U(\mathfrak{g})^0$.

(b) is proved similarly.

Let $F(\mathfrak{g}) = \{f \in \mathcal{U} U(\mathfrak{g})^0 | \dim_K U(\mathfrak{g}) : f < \infty\}$. Clearly $F(\mathfrak{g})$ is a subalgebra of $\mathcal{U} U(\mathfrak{g})^0$ and $f_0 \in F(\mathfrak{g})$.

**Theorem 5.1.** $F(\mathfrak{g})$ contains every finite dimensional simple $\mathfrak{g}$-module exactly once. Furthermore, $F(\mathfrak{g}) = \sum \mathcal{V}^\lambda$ a $\mathfrak{g}$-module direct sum over all $\lambda$ dominant integral (that is if $\alpha \in \mathfrak{h}^*$ then $2(\lambda, \alpha)/\langle \alpha, \alpha \rangle$ is a nonnegative integer). Furthermore $\mathcal{V}^\lambda : \mathcal{V}^{\lambda'} = \mathcal{V}^{\lambda + \lambda'}$ if $\lambda$ and $\lambda'$ are dominant integral.
Proof. Let \( V \) be a finite dimensional simple submodule of \( F(\mathfrak{g}) \). Then \( \dim_K V^{n+} = 1 \) and there is a dominant integral \( \lambda \in \mathfrak{h}^0 \) such that if \( v \in V^{n+} \) then \( h \cdot v = \lambda(h) \cdot v \). If \( g \in U(\mathfrak{g}) \), \( h \in \mathfrak{h} \), \( f \in V^{n+} \) then \( g = n^- h' n^+ \), \( n^- \in U(n^-) \), \( h' \in U(\mathfrak{h}) \), \( n^+ \in U(n^+) \) and \( f(h g) = f(h n^- h' n^+) = f(n^- h h' n^+) = 0 \) if \( n^- \neq 1 \) and \( n^+ \neq 1 \). Thus if \( n^- \) or \( n^+ \neq 1 \) then \( f(h g) = 0 = \lambda(h) f(g) \).

If \( n^- = n^+ = 1 \) then \( f(h g) = f(h h') = (h \cdot f)(h') = \lambda(h) f(h') \). Thus \( f \in T(\lambda)^{n+} = (V^\lambda)^{n+} \). And thus \( V = V^\lambda \).

If \( f \in F(\mathfrak{g}) \) then \( \dim_K U(\mathfrak{g}) \cdot f < \infty \). Thus \( U(\mathfrak{g}) \cdot f = \sum V_i \), \( V_i \) finite dimensional simple. Thus \( f \in \sum V^\lambda \). Hence \( F(\mathfrak{g}) = \sum V^\lambda \), \( \lambda \) dominant integral.

By Lemma 5.1 we have \( V^\lambda \cdot V^{\lambda'} \subset T(\lambda + \lambda') \). By the above the only finite dimensional simple submodule of \( T(\lambda + \lambda') \) is \( V^{\lambda + \lambda'} \). Thus if \( \lambda, \lambda' \) are dominant integral then \( V^\lambda \cdot V^{\lambda'} = V^{\lambda + \lambda'} \).

Let \( \pi = \{ \alpha_1, \ldots, \alpha_l \} \) and let \( \lambda_1, \ldots, \lambda_l \) be defined by \( 2(\alpha_i, \lambda_j)/\alpha_i \cdot \alpha_j = \delta_{ij} = 0 \) if \( i \neq j \), \( = 1 \) if \( i = j \). Then by Theorem 5.1 we know that \( V^{\lambda_1} + V^{\lambda_2} + \cdots + V^{\lambda_l} \) generates \( F(\mathfrak{g}) \). Let \( V = V^{\lambda_1} + V^{\lambda_2} + \cdots + V^{\lambda_l} \). Consider the symmetric algebra on \( V \), \( S(V) \), \( l \)-graded in the natural manner. That is \( S^n(V) \) where \( n = (n_1, \ldots, n_l) \), \( n_i \geq 0 \), \( n \) an integer is the space spanned by all products of \( n_1 \) elements of \( V^{\lambda_1} \), \( n_2 \) elements of \( V^{\lambda_2} \), etc. \( S(V) \) inherits a \( \mathfrak{g} \)-module structure from \( V \). And there is a natural algebra and \( \mathfrak{g} \)-homomorphism \( \Phi \colon S(V) \to F(\mathfrak{g}) \) such that \( S^n(V) \to V^{n_1 \lambda_1 + \cdots + n_l \lambda_l} \). Since \( F(\mathfrak{g}) \) is an integral domain \( \text{Ker} \Phi \) is a prime ideal in \( S(V) \). If \( I(V) = \{ f \in S(V) \mid x \cdot f = 0 \text{ for all } x \in \mathfrak{g} \} \) then \( I^+(V) = I(V) \cap_{n \neq 0} S^n(V) \) is a subalgebra of \( \text{Ker} \Phi \). We have not as yet found the relationship between \( \text{Ker} \Phi \) and \( I^+(V) \).

We conclude this section with a simple example of \( \text{Ker} \Phi \). Let \( \mathfrak{g} = A_2, \pi = \{ \alpha_1, \alpha_2 \} \). Then \( V^{\lambda} \) is just the \( 3 \) dimensional representation of \( A_2 \) as \( \text{sl}(3, K) \) and \( V^{\alpha} \) is just the dual module of \( V^{\alpha} \). If \( X_1 \in V^{\alpha_1}, X_2 \in V^{\alpha_2}, X_3 \in V^{\alpha_1 - \alpha_2} \). Identifying \( V^{\lambda} \) with the dual module of \( V^{\lambda} \) we take the dual basis \( Y_1, Y_2, Y_3, \) of \( X_1, X_2, X_3 \). \( (Y_1 \in V^{\alpha_1}, Y_2 \in V^{\alpha_2}, Y_3 \in V^{\alpha_1 - \alpha_2}) \) and \( \text{Ker} \Phi \) is generated by \( X_1 Y_1 + X_2 Y_2 + X_3 Y_3 \).

**Bibliography**


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