ON THE CENTRALIZER OF A LATTICE

S. P. WANG

Let $G$ be a locally compact group, $\Gamma$ a discrete subgroup and $G/\Gamma$ be the homogeneous space of left cosets. Let $\mu$ be a right Haar measure of $G$. $\mu$ induces a measure $\bar{\mu}$ over $G/\Gamma$. $\Gamma$ is called a lattice if $\bar{\mu}(G/\Gamma)$ is finite. By the density theorem, we know that when $G$ is a connected semisimple Lie group without compact factor, the centralizer of a lattice coincides with the center of $G$. In general, the centralizer of a lattice is not even abelian. In this short note, we shall prove the following theorem about the centralizer of a lattice in a Lie group.

**Theorem.** Let $G$ be a connected Lie group, $\Gamma$ a lattice and $Z(\Gamma)$ the centralizer of $\Gamma$ in $G$. Then the commutator subgroup $[Z(\Gamma), Z(\Gamma)]$ of $Z(\Gamma)$ is compact.

1. **Some lemmas.** Let $G$ be a Lie group, $\Gamma$ a discrete subgroup and $N(\Gamma)$ ($Z(\Gamma)$) the normalizer (centralizer) of $\Gamma$ in $G$. We shall establish some lemmas concerning $Z(\Gamma)$ which will be used later in proving the theorem.

**Lemma 1.1.** $N(\Gamma)$ is closed.

**Proof.** Let $x = \lim_n x_n$ such that $x_n \in N(\Gamma)$ for all $n$. Then $x_n \gamma x_n^{-1} \in \Gamma$ for all $n$, and $\gamma \in \Gamma$. Since $\Gamma$ is closed and $x\gamma x^{-1} = \lim_n x_n \gamma x_n^{-1}$, $x\gamma x^{-1} \in \Gamma$. It follows that $x\Gamma x^{-1} \subset \Gamma$. It is clear that $x^{-1} = \lim_n x_n^{-1}$ and $x_n^{-1} \in N(\Gamma)$ for all $n$. By the same argument, $x^{-1} \Gamma x \subset \Gamma$. Therefore $x \in N(\Gamma)$.

**Lemma 1.2.** $Z(\Gamma)\Gamma$ is closed.

**Proof.** Since $\Gamma$ is discrete, the identity component $N(\Gamma)^0$ of $N(\Gamma)$ is contained in $Z(\Gamma)$. Thus $Z(\Gamma)\Gamma$ is an open subgroup of $N(\Gamma)$. Hence it follows easily that $Z(\Gamma)\Gamma$ is closed.

Let $H$ be a locally compact and $\sigma$-compact group, $\Gamma$ a lattice, $K$ a closed subgroup containing $\Gamma$ and $\theta: H \rightarrow H/K$ be the projection map.

**Lemma 1.3.** If $\theta$ has a local cross section, then $\Gamma$ is a lattice of $K$.

**Proof.** Let $V$ be a compact neighborhood of $eK$ in $H/K$ and $s$ be a local cross section defined over $V$. Given any right Haar measure $\mu$.
of \( H \), we define \( \mu_v(B) = \mu(s(V)B) \) for any Borel subset \( B \) in \( K \). It is easy to verify that \( \mu_v \) is a right Haar measure of \( K \). Let \( F \) be a fundamental domain in \( K \) with respect to \( \Gamma \). Then \( s(V)F \) is a \( \Gamma \)-packing in \( G \), i.e. \( F^{-1}s(V)^{-1}s(V)F \cap \Gamma = \{ e \} \). Hence \( \mu_v(F) = \mu(s(V)F) \leq \mu(G/\Gamma) < \infty \) and \( \Gamma \) is a lattice of \( K \).

**Corollary 1.4.** Let \( G \) be a separable Lie group and \( \Gamma \) a lattice. If \( L \) is a closed subgroup containing \( \Gamma \), then \( \Gamma \) is a lattice of \( L \).

**Proof.** Immediate consequence of the existence of local cross section of the coset space.

**Proposition 1.5.** Let \( G \) be a connected Lie group, \( \Gamma \) a lattice. Then \( Z(\Gamma)/\text{center}(Z(\Gamma)) \) is compact.

**Proof.** By Lemma 1.2, \( Z(\Gamma)\Gamma \) is closed. Hence \( Z(\Gamma)/\Gamma \cap Z(\Gamma) \) is topologically isomorphic to \( Z(\Gamma)\Gamma/\Gamma \). By Corollary 1.4, \( \Gamma \) is a lattice of \( Z(\Gamma) \). Since \( \Gamma \) is normal in \( Z(\Gamma) \), \( Z(\Gamma)\Gamma/\Gamma \) is a Lie group with finite Haar measure. Therefore \( Z(\Gamma)/\Gamma \cap Z(\Gamma) \approx Z(\Gamma)\Gamma/\Gamma \) is compact. Clearly the center of \( Z(\Gamma) \) contains \( \Gamma \cap Z(\Gamma) \). It follows immediately that \( Z(\Gamma)/\text{center}(Z(\Gamma)) \) is compact.

2. **Generalization of a theorem of R. Baer.** Let \( G \) be a discrete group and \( Z(G) \) the center. It is well known that \( [G, G] \) is finite if \( [G: Z(G)] \) is finite. In this section, we shall generalize this result to Lie groups based on the discrete case.

**Proposition 2.** Let \( G \) be a Lie group such that \( G/Z(G) \) is compact. Then \( [G, G] \) is compact.

**Proof.** Let \( G^0 \) be the identity component of \( G \). It is easy to verify that \( G^0/Z(G) \cap G^0 \) is compact. Since center\((G^0)\) contains \( Z(G) \cap G^0 \), \( G^0/\text{center}(G^0) \) is compact. Therefore \( G^0 = K \times V \) where \( K \) is a connected compact Lie group and \( V \) a vector group. Hence \( [G^0, G^0] = [K, K] \) is compact. Consider \( G/[G^0, G^0] \); we may assume without loss of generality that \( G^0 \) is abelian. For fixed \( g \) in \( G \), let \( \theta_g : G^0 \to G^0 \) be the map defined by \( \theta_g(g_0) = gg_0g^{-1}g^{-1} \), \( g_0 \in G^0 \). Since \( G^0 \) is normal and abelian, \( \theta_g \) is a continuous homomorphism. As \( G^0/Z(G) \cap G^0 \) is compact, \( \text{Im}(\theta_g) \) is compact. Let \( H \) be a subgroup generated by finitely many subgroups \( \text{Im}(\theta_{g_1}), \ldots, \text{Im}(\theta_{g_m}) \) such that \( H \) is of maximum dimension among all choices of finite set \( \{ g_1, \ldots, g_m \} \). Then one verifies without difficulty that \( [G, G^0] = H \) is compact. Consider \( G/[G, G^0] \); we may assume without loss of generality that \( G^0 \) is central, i.e. \( G^0 \subset Z(G) \). Hence we may assume that \( G/Z(G) \) is discrete and compact, i.e. finite. However in the final case, the proposition is well known. Therefore \( [G, G] \) is compact.
3. **Main result and its applications.** Let $G$ be a connected Lie group and $\Gamma$ a lattice. By Proposition 1.5, $Z(\Gamma)/\text{center}(Z(\Gamma))$ is compact. By Proposition 2, $[Z(\Gamma), Z(\Gamma)]$ is compact. This completes the proof of our main theorem.

As an immediate consequence of the theorem, we get the following corollaries.

**Corollary 3.1.** Let $G$ be a connected Lie group such that $\{e\}$ is the only maximum compact subgroup. If $\Gamma$ is a lattice of $G$, then $Z(\Gamma)$ is abelian.

**Corollary 3.2.** Let $G$ be a simply connected solvable Lie group and $\Gamma$ a lattice. Then $Z(\Gamma)$ is abelian.

**Corollary 3.3.** Let $G$ be a connected Lie group and $R(G)$ its radical such that $R(G)$ is simply connected and $G/R(G)$ without compact factor. If $\Gamma$ is a lattice of $G$, then $Z(\Gamma)$ is abelian.

**References**


**Institute for Advanced Study**