DECOMPOSITION THEOREMS FOR VECTOR MEASURES

JAMES K. BROOKS

1. Introduction. In this note we consider two theorems for vector measures, each of which is a generalization of a well-known theorem for scalar measures. Theorem 1 extends the Yosida-Hewitt theorem [7], [4, p. 163], which states that a bounded real valued finitely additive (f.a.) measure defined on a Boolean algebra can be decomposed uniquely into a countably additive (c.a.) part and a purely finitely additive (p.f.a.) part. The second theorem, due to Rickart [6, Theorem 4.5], is a Lebesgue decomposition theorem for c.a. vector measures with respect to an outer measure. Several authors have given alternate proofs of a restricted form of this theorem (e.g. see [3, p. 189] where the outer measure is replaced by a measure and the total variation of the vector measure is assumed to be finite—a condition not generally satisfied). In 3 we give a short and elementary proof of Rickart’s theorem, which represents a considerable simplification of the existing proofs of this result.

Let $X$ be a Banach space over the reals $\mathbb{R}$ with first and second conjugate spaces $X^*$ and $X^{**}$; we regard $X$ as a subset of $X^{**}$. $\Sigma_0$ and $\Sigma$ respectively denote a Boolean algebra and a $\sigma$-algebra of subsets of a set $S$. $\|\mu\|$ denotes the semivariation [3, p. 51] of $\mu$. $\mu$ is bounded if $\|\mu\|(S) < \infty$.

2. Theorem 1. Let $\mu: \Sigma_0 \rightarrow X$ be bounded and finitely additive. Then $\mu$ can be written uniquely in the form $\mu = \mu_1 + \mu_2$, where the $\mu_1: \Sigma_0 \rightarrow X^{**}$ are finitely additive and for each $f \in X^*$: (1) $\mu_1(\cdot)f: \Sigma_0 \rightarrow \mathbb{R}$ is countably additive and (2) $\mu_2(\cdot)f: \Sigma_0 \rightarrow \mathbb{R}$ is purely finitely additive.

Proof. For $f \in X^*$ the set function $f \mu$ defined by $(f \mu)(E) = f(\mu(E))$, $E \in \Sigma_0$ is f.a. Moreover, since $|f \mu(E)| \leq \|f\| \|\mu(E)\| \leq \|f\| \|\mu\|(S)$, $f \mu$ is bounded. By the Yosida-Hewitt theorem $f \mu = f \mu_1 + f \mu_2$, where $\mu_1$ is c.a. and $\mu_2$ is p.f.a. For $E \in \Sigma_0$ define $F_{i,B}: X^* \rightarrow \mathbb{R}$, $i = 1, 2$, as follows: $F_{i,B}(f) = \mu_i(\cdot)(E)$, $f \in X^*$. It follows from the uniqueness of the decomposition of the scalar measures and Theorem 1.17 in [7] that $\mu_{cf,i} = \alpha \mu_{f,i}$ and $\mu_{f+g,i} = \mu_{f,i} + \mu_{g,i}$, for $f, g \in X^*$, $\alpha \in \mathbb{R}$; hence $F_{i,B}$ is linear on $X^*$. To show that $F_{i,B} \in X^{**}$, let $f \in X^*$ and let $f \mu = (f \mu)^+ - (f \mu)^-$, $|f \mu| = (f \mu)^+ + (f \mu)^-$ be the Jordan decomposition of $f \mu$. Again using the uniqueness of the decomposition of $f \mu$, we have $u_{f,i} = (f \mu)^+ - (f \mu)^-$ where the subscripts refer to the c.a. and p.f.a. parts. $|F_{i,B}(f)|$

Received by the editors April 10, 1968.

27
\[\mu_\lambda(E) = (f\mu_\lambda)^+(E) + (f\mu_\lambda)^-(E) \leq 2\sup_{A \in B} |f\mu(A)| \leq 2\sup_{A \in B} \|f\| \|\mu(A)\| \leq 2\|f\| \|\mu\|(E).\]

Thus \[\|F_{1,B}\| \leq 2\|\mu\|(E)\] and \[F_{1,B} \in X^{**}.\]
Define \[\mu_1(E) = F_{1,B}, E \in \mathcal{S}_0.\] It is clear from the construction that \(\mu_1\) and \(\mu_2\) have the required properties.

**Corollary.** Assume \(X\) is reflexive and \(\mu : \Sigma \to X\) is bounded and finitely additive. Then \(\mu\) can be written uniquely in the form \(\mu = \mu_1 + \mu_2, \mu_i : \Sigma \to X,\) where (1') \(\mu_1\) is countably additive and (2') \(f\mu_2\) is purely finitely additive for each \(f \in X^*.\)

**Proof.** Let \(\mu_i\) be the components of \(\mu\) given in the above theorem. Since \(f\mu_i(\cdot) = \mu_i(\cdot)f\) is c.a. for \(f \in X^*, \mu_1\) is weakly countably additive in addition to being f.a. By the Pettis theorem [4, p. 318] \(\mu_1\) is c.a., and the result follows.

**Remark.** For a reflexive space \(X,\) the existence of a large number of bounded f.a. \(X\)-valued measures on \(\Sigma,\) hence measures satisfying (2'), can be deduced as follows. Let \(Y = M_{X^*}(\Sigma)\) be the Banach space of totally measurable \(X^*\)-valued functions (i.e. uniform limits of simple functions) with the uniform norm. Since \(Y^*\) is isomorphic to the set of all f.a. \(X\)-measures on \(\Sigma\) having finite total variation [2], the Hahn-Banach theorem implies that for each \(0 \neq g \in Y\) and \(\alpha \in \mathbb{R},\) there exists a f.a. \(X\)-measure \(\mu\) such that \(\int g d\mu = \alpha,\) where the integral is defined in [3].

### 3. The Lebesgue decomposition

**Definitions.** Let \(\beta\) be an outer measure defined on \(\Sigma.\) Then \(\mu : \Sigma \to X\) is \(\beta\)-continuous if \(\beta(E) \to 0\) implies \(\mu(E) \to 0.\) \(\mu\) is \(\beta\)-singular if there exists an \(E^* \in \Sigma\) such that \(\beta(E^*) = 0\) and \(\mu(E) = \mu(E \cap E^*), E \in \Sigma.\)

**Theorem 2.** Let \(\mu : \Sigma \to X\) be countably additive and let \(\beta\) be an outer measure defined on \(\Sigma.\) Then \(\mu\) can be decomposed uniquely into the form \(\mu = \mu_1 + \mu_2,\) where \(\mu_1\) and \(\mu_2\) are countably additive, \(\mu_1\) is \(\beta\)-continuous, and \(\mu_2\) is \(\beta\)-singular.

**Proof.** The uniqueness of \(\mu_1\) and \(\mu_2\) is obvious. By a theorem due to Bartle, Dunford, and Schwartz [1], there exists a finite nonnegative c.a. measure \(\lambda\) defined on \(\Sigma\) such that \(\mu = \lambda\)-continuous (see [5] for an elementary proof of this result). \(\lambda\) can be decomposed uniquely into a \(\beta\)-continuous part \(\lambda_c\) and a \(\beta\)-singular part \(\lambda_s.\) This can be seen by examining the proof of the classical Lebesgue decomposition theorem in [4, p. 132]. The argument used there remains valid if the appropriate measure appearing in the proof is replaced by an outer measure. Consequently, there exists a set \(E^* \in \Sigma\) such that \(\beta(E^*) = 0\) and \(\lambda_c(E) = \lambda(E \cap E^*), \lambda_c(E) = \lambda(E \cap (S - E^*)).\) Let \(\mu_1(E)\)
\( = \mu(E \cap (S - E^*)) \) and \( \mu_2(E) = \mu(E \cap E^*) \). Obviously \( \mu_2 \) is \( \beta \)-singular. Now if \( \beta(E_n) \to 0 \), then \( \lambda(E_n \cap (S - E^*)) = \lambda(S - E^*) \to 0 \); hence \( \mu_1(E_n) = \mu(E_n \cap (S - E^*)) \to 0 \), and \( \mu_1 \) is \( \beta \)-continuous. The conclusion of the theorem now follows.

Added in proof. The classical Lebesgue decomposition theorem (even with respect to an outer measure) can be proved in a straightforward fashion that avoids the Radon-Nikodym theorem. This method was used by the author to decompose set functions of a more general type than the ones considered here (cf. An integration theory for set-valued measures. I, Bull. Soc. Roy. Sci. Liège, no5-8 (1968), 312–319). We shall sketch the proof. Let \( \lambda \) and \( \beta \) be as in the above proof. Define \( \mathfrak{M} = \{ E \in \Sigma : \beta(E) = 0 \} \); \( \eta = \sup \lambda(E), E \in \mathfrak{M} \). By using the method of exhaustion, one can construct a set \( E^* \) such that \( E^* \in \mathfrak{M} \) and \( \lambda(E^*) = \eta \). It follows that if \( E \in \mathfrak{M} \), then \( \lambda(E - E^*) = 0 \). Then define \( \lambda_1 \) and \( \lambda_2 \) to be the restrictions of \( \lambda \) to \( E^* \) and \( S - E^* \) respectively.

Bibliography


University of Florida