SHEAF COHOMOLOGY WITH BOUNDS AND BOUNDED HOLOMORPHIC FUNCTIONS

YUM-TONG SIU

Suppose \( U \) is the unit disc in \( \mathbb{C} \). For \( 0 < r < 1 \) \( Q_r \) (or simply \( Q \)) is the annulus \( \{ z \in U \mid |z| > r \} \). A subvariety \( V \) of pure codimension 1 in \( U^N \) is called a Rudin subvariety if for some \( r \) \( V \cap Q^r = \emptyset \). A Rudin subvariety is called a special Rudin subvariety if there is \( \delta > 0 \) such that, for \( 1 \leq k \leq N \), \( (a', a_i, a'') \in (Q^{k-1} \times U \times Q^{N-k}) \cap V \), \( i = 1, 2, \) and \( a_i \neq a_j \), we have \( |a_i - a_j| \geq \delta \). If a holomorphic function \( f \) generates the ideal-sheaf of its zero-set \( E \), then we write \( Z(f) = E \). The Banach space of all bounded holomorphic functions on a reduced complex space \( X \) under the sup norm is denoted by \( H^\infty(X) \) and the norm of \( f \in H^\infty(X) \) is denoted by \( \| f \|_X \). The following two theorems were proved by W. Rudin [2] and H. Alexander [1] respectively.

**Theorem 1.** If \( V \) is a Rudin subvariety, then there is \( f \in H^\infty(U^N) \) such that \( Z(f) = V \).

**Theorem 2.** If \( V \) is a special Rudin subvariety, then there is a bounded linear map from \( H^\infty(V) \) to \( H^\infty(U^N) \) which extends every bounded holomorphic function on \( V \) to one on \( U^N \).

Cartan's Theorem B implies that an analytic hypersurface of a polydisc is the zero-set of a holomorphic function and that every holomorphic function on the hypersurface is induced by a holomorphic function on the polydisc. One can expect that some Theorem B with bounds would easily yield the above two theorems. In this note we prove a simple theorem on sheaf cohomology with bounds (Theorem 3 below) which can imply Theorems 1 and 2. This gives us more perspective proofs of these two theorems.

Suppose \( X \) is a reduced complex space and \( \mathcal{O} \) is the structure-sheaf of \( X \times U^N \). Let \( W_k = X \times U^{k-1} \times Q \times U^{N-k}, 1 \leq k \leq N, \) and \( \mathcal{W} = \{ W_k \} \). For \( \nu \geq 0 \) and \( 1 \leq i_0, \ldots, i_\nu \leq N \) \( W_{i_0} \cap \cdots \cap W_{i_\nu} \) denotes \( W_{i_0} \cap \cdots \cap W_{i_\nu} \). If \( f \in C^\infty(\mathcal{W}, \mathcal{O}) \), then \( f_{i_0, \ldots, i_\nu} \in \Gamma(W_{i_0} \cap \cdots \cap W_{i_\nu}, \mathcal{O}) \) denotes the value of \( f \) at the simplex \( (W_{i_0}, \ldots, W_{i_\nu}) \) of the nerve of \( \mathcal{W} \). Let \( \rho = 2/(1 - r) \) and for \( 1 \leq \nu < N \) let

\[
\sigma_\nu = \sum_{\mu=1}^{N-\nu} \binom{N}{\mu} (\nu + 1)^{\nu-1} \rho^\mu.
\]

Received by the editors March 11, 1968.
**Lemma 1.** Suppose $f$ is a bounded holomorphic function on $X \times Q$ whose absolute value is bounded by a positive number $K$. Suppose for $w \in X f(w, z) = \sum_{\mu > 0} h_{\mu}(w)z^\mu$ is the Laurent series expansion of $f$ in $z$ (where $z$ is the coordinate function of $Q$). Let $g(w, z) = \sum_{\mu > 0} h_{\mu}(w)z^\mu$ on $X \times Q$. Then $\|g\|_{X \times Q} \leq K$.

**Proof.** Fix $(w, z) \in X \times Q$. Choose arbitrarily two positive numbers $a$ and $b$ such that $r < a < |z| < b < 1$. We need only prove that $|g(w, z)| \leq 2bK/(b-a)$, because the result follows then from letting $a \to r$ and $b \to 1$.

**Case (i).** $|z| \leq (a+b)/2$. Then $|f-z| \leq (b-a)/2$ for $|f| = b$.

\[
|g(w, z)| = \left| \frac{1}{2\pi i} \int_{|z|=b} \frac{f(w, \xi)}{\xi - z} \, d\xi \right| \leq \frac{2b}{b-a} K.
\]

**Case (ii).** $|z| \geq (a+b)/2$. Then $|f-z| \leq (b-a)/2$ for $|f| = a$.

\[
|f(w, z) - g(w, z)| = \left| \frac{1}{2\pi i} \int_{|\xi|=a} \frac{f(w, \xi)}{\xi - z} \, d\xi \right| \leq \frac{2a}{b-a} K.
\]

Hence $|g(w, z)| \leq 2bK/(b-a)$. Q.E.D.

**Theorem 3.** For $1 \leq \nu < N$ there exists a linear map $\phi: B^r(\mathcal{M}, \theta) \to C^{r-1}(\mathcal{B}, \theta)$ over the ring of all holomorphic functions on $X$ such that

(i) $\phi\theta = \text{the identity map on } B^r(\mathcal{M}, \theta)$, and

(ii) if $f \in B^r(\mathcal{M}, \theta)$ and $\|f_{i_0, \ldots, i_{\nu-1}}\|_{W_{i_0, \ldots, i_{\nu-1}}} \leq K$ for $1 \leq i_0, \ldots, i_{\nu-1} \leq N$, then $\|\phi(f)_{i_0, \ldots, i_{\nu-1}}\|_{W_{i_0, \ldots, i_{\nu-1}}} \leq K$.

**Proof.** First we define for $1 \leq i \leq N$ and $0 \leq \nu < N$ a linear map $e_i: C^r(\mathcal{M}, \theta) \to C^{r-1}(\mathcal{B}, \theta)$ over the ring of all holomorphic functions on $X$ as follows: Suppose $f \in C^r(\mathcal{M}, \theta)$. If $f_{i_0, \ldots, i_{\nu-1}} = \sum_{\mu > 0} h_{\mu} z_i^\mu$ is the Laurent series expansion of $f_{i_0, \ldots, i_{\nu-1}}$ in $z_i$ (where $z_i$ is the $i$th coordinate function of $U_i$), then $(e_i(f))_{i_0, \ldots, i_{\nu-1}} = \sum_{\mu > 0} h_{\mu} z_i^\mu$. By applying Lemma 1 with $X$ replaced by the product of $X$ and $U_i$, we have $\|e_i(f)_{i_0, \ldots, i_{\nu-1}}\|_{W_{i_0, \ldots, i_{\nu-1}}} \leq K$ for $1 \leq i_0, \ldots, i_{\nu-1} \leq N$. Observe that $(1 - e_i)(f)_{i_0, \ldots, i_{\nu-1}} = 0$ if $i \neq i_0, \ldots, i_\nu$. For $0 \leq \nu < N - 1$ we have $(1 - e_i) \circ (1 - e_{i_\nu}) \circ \cdots \circ (1 - e_{i_0}) = 0$ on $C^{r-1}(\mathcal{M}, \theta)$, because for any $1 \leq i_0, \ldots, i_\nu \leq N$ there exists $1 \leq i \leq N$ such that $i \neq i_0, \ldots, i_\nu$. Since $e_i$ commutes with $\theta$, for $1 \leq \nu < N$ we have $1 = (1 - e_i) \circ (1 - e_{i_\nu}) \circ \cdots \circ (1 - e_{i_0}) = 0$ on $B^r(\mathcal{M}, \theta)$.

Next we define for $1 \leq i \leq N$ and $1 \leq \nu < N$ a linear map $k_i: C^r(\mathcal{M}, \theta) \to C^{r-1}(\mathcal{B}, \theta)$ over the ring of all holomorphic functions on $X$ as follows: If $f \in C^r(\mathcal{M}, \theta)$, then set $(k_i(f))_{i_0, \ldots, i_{\nu-1}}$ to be the holomorphic function on $W_{i_0, \ldots, i_{\nu-1}}$ whose restriction to $W_{i_0, \ldots, i_{\nu-1}}$ is $(e_i(f))_{i_0, \ldots, i_{\nu-1}}$. Straightforward computation shows that for $1 \leq i \leq N$ and $1 \leq \nu < N$ we have $e_i = \delta k_i - k_i \delta$ on $C^r(\mathcal{M}, \theta)$. Hence for $1 \leq \nu < N$ we have $(1 - \delta k_i) \circ (1 - \delta k_{i_\nu}) \circ \cdots \circ (1 - \delta k_0) = 0$ on $B^r(\mathcal{M}, \theta)$. For $1 \leq \nu < N$...
define \( \phi_r : B^r(\mathbb{R}, \emptyset) \to C^{-1}(\mathbb{R}, \emptyset) \) by
\[
\phi_r = \sum_{\mu=1}^{N} (-1)^{n-1} \sum_{i_1 < \ldots < i_n} k_{i_1} \delta k_{i_2} \cdots \delta k_{i_n}.
\]
Then \( \phi_r \) satisfies the requirement. Q.E.D.

**Remark.** By using \( \sup |\text{Re}\phi(f)_{i_3} \cdots_{i_m}| \) on \( W_{i_3} \cdots_{i_m} \) instead of using \( \|\phi(f)_{i_3} \cdots_{i_m}\|_{W_{i_3} \cdots_{i_m}} \), a theorem similar to Theorem 3 can be proved. We need only prove a lemma which corresponds to Lemma 1 but uses sup norms of the real parts instead. To do this, we observe that \( f \mapsto \text{Re} \, f \) defines a continuous \( R \)-linear injection with closed image from the Fréchet space \( E \) of all holomorphic functions on \( Q \) whose constant coefficients in the Laurent series expansions are real to the Fréchet space of all harmonic functions on \( Q \). Hence, for \( r < a < b < 1 \), there exists a constant \( C \) such that, if \( f \in E \) and \( \sup |\text{Re}\, f| \leq K \), then \( |f(z)| \leq CK \) on \( a \leq |z| \leq b \). The desired lemma follows from an argument analogous to the proof of Lemma 1, but this time we leave \( a \) and \( b \) fixed instead of letting \( a \to r \) and \( b \to 1 \) and do not restrict \( |z| \) to \((a, b)\).

**Proof of Theorem 1.** By Cartan's Theorem B there is a holomorphic function \( \tilde{f} \) on \( U^N \) such that \( Z(\tilde{f}) = V \). We can assume \( V \cap (Q_r)^N \neq \emptyset \) for some \( r' < r \). We are going to prove (1)* by induction on \( k \).

On \( U^k \times Q^{N-k} \) (and likewise on products obtained by permuting the \( N \) factors) we can construct a bounded holomorphic function \( f^{(k)} \) such that \( Z(f^{(k)}) = (U^k \times Q^{N-k}) \cap V \) and \( (f^{(k)})^{-1} \) is bounded on \( Q^N \).

\( Q^N \cap V = \emptyset \) implies that \( (U \times Q^{N-1}) \cap V \) is an analytic cover over \( Q^{N-1} \) of, say, \( n \) sheets. There exists a proper subvariety \( A \) in \( Q^{N-1} \) and locally defined holomorphic functions \( g^{(1)}, \ldots, g^{(n)} \) on \( Q^{N-1} \) such that \( (U \times (Q^{N-1} - A)) \cap V = \{(z_1, \ldots, z_N) \in U \times (Q^{N-1} - A) | z_1 = g^{(i)}(z_2, \ldots, z_N) \text{ for some } i \} \). The bounded holomorphic extension \( f^{(1)} \) on \( U \times Q^{N-1} \) of \( \prod_{i=1}^{n} (z_i - g^{(i)}(z_2, \ldots, z_N)) \) satisfies \( Z(f^{(1)}) = (U \times Q^{N-1}) \cap V \) and \((f^{(1)})^{-1} \) is bounded on \( Q^N \). (1)* is proved. Suppose (1)* is true for \( 1 \leq k < m \). Then for \( 1 \leq i \leq m \) we can construct a bounded holomorphic function \( f_i \) on \( G_i = U^{i-1} \times Q \times U^{m-i} \times Q^{N-m} \) such that \( Z(f_i) = G_i \cap V \) and \( f_i^{-1} \) is bounded on \( Q^N \). By replacing \( f_i \) by the product of \( f_i \) with suitable powers of \( z_{i}, z_{m+1}, \ldots, z_N \), we can assume that we can select a regular branch \( h_i \) of \( \log(f_i/f_i) \) on \( G_i \). Since \( h_i - h_j = \log(f_j/f_i) \) has bounded real part on \( G_i \cap G_j \), by the Remark following Theorem 3 we can construct holomorphic functions...
\( h_i \) on \( G_i \) with bounded real parts such that \( h_i - h_j = a \) branch of \( \log(f_i/f_j) \). The holomorphic function \( f^{(m)} \) on \( U^m \times Q^{N-m} \) which agrees with \( f_i \), \( \exp(h_i) \) on \( G_i \) satisfies \( Z(f^{(m)}) = (U^m \times Q^{N-m}) \cap V \) and is bounded. Moreover, \( (f^{(m)})^{-1} \) is bounded on \( Q^N \). (1)_m is proved. The theorem follows from (1). Q.E.D.

Proof of Theorem 2. By Theorem 1 we can construct \( g \in H^\omega(U^n) \) such that \( Z(g) = V \). The construction implies that \( g^{-1} \) is bounded on \( Q^N \). Take \( f \in H^\omega(V) \). By Cartan's Theorem B, \( f \) is the restriction to \( V \) of a holomorphic function \( f \) on \( U^n \). We are going to prove (2)_k by induction on \( k \).

On \( U^k \times Q^{N-k} \) (and likewise on products obtained by permuting the \( N \) factors) we can construct a bounded holomorphic function \( f^{(k)} \) which agrees with \( f \) on \( (U^k \times Q^{N-k}) \cap V \).

From the conditions of special Rudin subvarieties we conclude that \( (U^m \times Q^{N-m}) \cap V \) is an unbranched analytic cover over \( Q^{N-1} \) of, say, \( n \) sheets. There are locally defined holomorphic functions \( g^{(i)} \), \( \cdots \), \( g^{(n)} \) on \( Q^{N-1} \) such that \( (U^m \times Q^{N-m}) \cap V = \{(z_1, \ldots, z_N) \in U^m \times Q^{N-1} | z_i = g^{(i)}(z_2, \ldots, z_N) \text{ for some } i \} \). The function \( f^{(1)}(z_1, \ldots, z_N) = \sum_{i=1}^n f_i(g^{(i)}(z_2, \ldots, z_N), z_2, \ldots, z_N) - g^{(i)}(z_2, \ldots, z_N) \) is well defined, agrees with \( f \) on \( (U^m \times Q^{N-m}) \cap V \), and is bounded. (2)_1 is proved. Suppose (2)_k is true for \( 1 \leq k < m \). We can construct bounded holomorphic functions \( f_i \) on \( G_i = U_i^1 \times Q \times U^m-i \times Q^{N-m} \), \( 1 \leq i \leq m \), such that \( f_i = f \) on \( G_i \cap V \). Let \( h_i = (f_i - f_j)/g \) on \( G_i \). Since \( h_i - h_j = (f_i - f_j)/g \) is bounded on \( G_i \cap G_j \) (because \( g^{-1} \) is bounded on \( Q^N \)), we can construct by Theorem 3 \( h_i \in H^\omega(G_i) \) such that \( h_i - h_j = h_i - h_j = (f_i - f_j)/g \). The holomorphic function \( f^{(m)} \) on \( U^m \times Q^{N-m} \) which agrees with \( f_i + gh_i \) on \( G_i \) is bounded and agrees with \( f \) on \( (U^m \times Q^{N-m}) \cap V \). (2)_m is proved. By (2)_N we can construct \( f^{(N)} \in H^\omega(U^N) \) which agrees with \( f \) on \( V \). It is clear from the constructions that the map defined by \( f \mapsto f^{(N)} \) is a bounded linear map from \( H^\omega(V) \) to \( H^\omega(U^N) \). Q.E.D.

The author wishes to thank H. Alexander for pointing out some errors in an earlier version of this note.

References


University of Notre Dame