REMARK ON A PAPER OF Y. IKEBE

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In a recent paper, Y. Ikebe [3] has proved a theorem on characterization of best approximations by elements of linear subspaces in the space of all continuous real- or complex-valued functions on a compact space. In the present paper we want to observe that this result can be extended to arbitrary real or complex normed linear spaces. We shall use the notations of [5].

THEOREM. Let $E$ be a normed linear space, $G$ a linear subspace of $E$ and $x \in E \setminus G$. An element $g_0 \in G$ is a best approximation of $x$ (i.e. $\|x - g_0\| = \inf_{g \in G} \|x - g\|$) if and only if $0$ belongs to the $\sigma(G^*, G)$-closure of the convex hull of the set

$$(1) \quad A = \{ |f(x - g_0)| \} \cdot f \in E(S_{E^*}), \; |f(x - g_0)| = \|x - g_0\| \},$$

where $E(S_{E^*})$ denotes the set of all extreme points of the unit cell $S_{E^*} = \{ f \in E^* \mid \|f\| \leq 1 \}$, $f|_G$ denotes the restriction of $f$ to the subspace $G$ and where $[\cdot ]^*$ stands for complex conjugate.

Proof. Let us denote by $\Omega(A)$ the $\sigma(G^*, G)$-closure of the convex hull of $A$.

Necessity. Assume that $0 \in \Omega(A)$. Then, as observed also in [3], there exists an element $g_1 \in G$ such that

$$\sup_{\phi \in \Omega(A)} \text{Re} \phi(g_1) < 0. \quad (2)$$

Now, if $g_0$ would be a best approximation of $x$, then, by [5, pp. 57–58, Corollary 1.9], for every $g \in G$ there would exist an $f^* \in E(S_{E^*})$ such that

$$|f^*(x - g_0)| = \|x - g_0\|, \quad \text{Re} (|f^*(x - g_0)| - f^*(g)) \geq 0, \quad (3)$$

whence, for $\phi^* = [f^*(x - g_0)] f^*|_G$ one would obtain

$$\phi^* \in A, \quad \text{Re} \phi^*(g) \geq 0, \quad (4)$$

which contradicts (2).

Sufficiency can be proved similarly to [3]: assuming that $0 \in \Omega(A)$ and using the $\sigma(E^*, E)$-compactness of $S_{E^*}$, one easily obtains an $f \in E^*$ such that

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\[ \|f\| = 1, \quad f(g) = 0 \quad (g \in G), \]

whence \([4]\) \(g_0\) is a best approximation of \(x\). This completes the proof of the theorem.

In the particular case when \(E = \mathbb{C}(Q)\), the space of all continuous real- or complex-valued functions on a compact space \(Q\), we have \(f \in \mathcal{E}(S_{g^*})\) if and only if \(f\) is of the form \(\alpha_0 e_{g_0}\), i.e.

\[ f(y) = \alpha_0 y(g_0) \quad (y \in \mathbb{C}(Q)), \]

where \(\alpha_0\) is a scalar with \(|\alpha_0| = 1\) and \(g_0 \in Q\) (see e.g. [2, p. 441, Lemma 6], or, for another proof, [5, p. 73]). Therefore

\[ A = \{ \langle \alpha(x(q) - g_0(q)) \rangle | \alpha | = 1, \quad q \in Q, \}

\]

and thus the above theorem yields the result of Y. Ikebe [3].

In the particular case when \(\dim G = n < \infty\), the corollary given in [3] with reference to E. W. Cheney [1] can be also extended to arbitrary normed linear spaces as follows:

**Corollary.** Let \(E\) be a normed linear space, \(G\) a linear subspace of \(E\) with \(\dim G = n < \infty\), and \(x \in E \setminus G\). An element \(g_0 \in G\) is a best approximation of \(x\) if and only if \(0\) belongs to the convex hull of the set (in the \(n\)-space)

\[ B = \{ \langle f(x - g_0) - f(x_1), \cdots, f(x - g_0) - f(x_n) \rangle | f \in \mathcal{E}(S_{g^*}), \]

\]

where \(x_1, \cdots, x_n \in G\) is some basis of \(G\).

This corollary can be deduced from the above theorem as in the corresponding particular case in [3], observing that if \(\dim G < \infty\), then the convex hull of the set \(A\) defined by (1) is \(\sigma(G^*, G)\)-closed and using the isomorphic mapping \(\phi \rightarrow (\phi(x_1), \cdots, \phi(x_n))\) of \(G^*\) onto the \(n\)-space.

**References**

