1. Introduction. In [1] Foguel provides a counterexample to Nagy's question of whether every power bounded operator on a Hilbert space is similar to a contraction. Generally questions about powers of operators have exponential analogues which can be phrased as questions about \((C_0)\) semigroups. The purpose of this note is to provide a counterexample to the semigroup analogue of Nagy's question—that is, to construct a uniformly bounded \((C_0)\) semigroup on a Hilbert Space \(H\) whose generator is not similar to a dissipative operator. (An operator \(L\) is dissipative on \(H\) if \(\langle Ly, y \rangle + \langle y, Ly \rangle \leq 0\) for all \(y \in \text{Domain } (L)\).) The method used depends strongly on Foguel's ideas and utilizes the viewpoint presented in Halmos' note [2] on Foguel's counterexample.

2. A necessary condition for similarity to a dissipative operator. Let \(S(t)\) denote a \((C_0)\) semigroup and \(L\) its infinitesimal generator on \(H\). We will need the characterization of dissipative semigroup generators presented by Phillips [3, p. 203] which says that \(L\) is dissipative if and only if \(\|S(t)\| \leq 1\) for all \(t \geq 0\).

Paralleling Foguel we define

\[ W(L) = \{ x \in H : \text{weak limit } (t \to \infty) S(t)x = 0 \}. \]

As a necessary condition for similarity to a dissipative operator we prove:

**Lemma 1.** If \(L\) is similar to a dissipative operator, then \(W(L) \cap [W(L^*)]^\perp = \{0\} \).

**Proof.** The argument is broken down into three parts in the manner of Halmos.

*Part I.* If \(A\) generates a group of unitary operators \(U(t), -\infty < t < \infty\), then \(W(A) = W(A^*)\) where \(A^*\) generates the group \(U^*(t)\). To see this, represent \(U(t)\) as multiplication by \(u^t\) on some \(L_2(\mu)\) isometrically isomorphic to \(H\), where \(u\) is a measurable function of constant modulus one. Showing \(W(A) \subseteq W(A^*)\) means showing that
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$\int u'f'g'd\mu \to 0$ for all $g \in L_2(\mu)$ implies $\int u'f'h'd\mu \to 0$ for all $h \in L_2(\mu)$. This follows by setting, for each $h$, $g = (\text{sgn} f)^* h$ and taking complex conjugates. By symmetry $W(A^*) \subseteq W(A)$, so equality holds.

**Part II.** If $D$ is dissipative and hence generates a contraction semigroup $C(t)$, then $W(D) = W(D^*)$. For this argument, the theory of strong unitary dilations [4, p. 15] yields a strongly continuous group $U(t)$ of unitary operators on a Hilbert space $H' \supset H$ such that $PU(t)u = C(t)$ for all $t \geq 0$ where $P$ is the orthogonal projection of $H'$ onto $H$. Furthermore $H'$ is minimal in the sense that it is spanned by the elements $\{ U(t)x : x \in H \}$ and $-\infty < t < \infty$. Define $H_s = \{ y \in H' : \langle U(t)x, y \rangle \to 0 \text{ as } t \to \infty \}$. For $x \in W(D)$, $\langle C(t)x, y \rangle = \langle PU(t)x, y \rangle = \langle U(t)x, Py \rangle = \langle U(t)x, y \rangle \to 0$ as $t \to \infty$ for all $y \in H$. Hence $H \subseteq H_s$ whenever $x \in W(D)$. Furthermore, $H_s$ is a linear manifold invariant under $U(t)$ and $U^*(t)$, $-\infty < t < \infty$, and since $H'$ is minimal as stated above, it follows that $x \in W(D)$ implies $H_s = H'$. Call $A$ the generator of $U(t)$ and note that we then have $W(D) \subseteq W(A) \cap H$ where $W(A)$ is defined on $H'$. Conversely, $x \in W(A) \cap H$ implies $\langle U(t)x, y \rangle \to 0$ as $t \to \infty$ for all $y \in H'$. Thus for $y \in H$, $\langle (C(t)x, y) = \langle PU(t)x, y \rangle = \langle U(t)x, Py \rangle = \langle U(t)x, y \rangle \to 0$ as $t \to \infty$, so that $x \in W(D)$. This shows that $W(A) \cap H = W(D)$ and using the same fact for $A^*$ and $D^*$ and Part I, $W(D) = W(D^*)$ follows.

**Part III.** The completion of the lemma now follows by noting that if $L = SDS^{-1}$, $D$ dissipative, then $W(L) = S(W(D))$ and $[W(L^*)]^\perp = S([W(D^*)]^\perp)$. Since, by Part II, $W(D) = W(D^*)$ so that $W(D) \cap [W(D^*)]^\perp = \{ 0 \}$, we obtain $W(L) \cap [W(L^*)]^\perp = \{ 0 \}$.

3. **Construction of the counterexample.** Let $H$ be the Hilbert space $L_2([0, \infty))$ and let $V(t)$ be right translation by $t$ on $H$, so for $t \geq 0$,

$$V(t)f(x) = f(x - t) \quad \text{when } x \geq t,$$

$$= 0 \quad \text{otherwise},$$

and

$$V^*(t)f(x) = f(x + t).$$

It is easy to check that $V(t)$ and $V^*(t)$ are $(C_0)$ semigroups of operators whose norms are identically one.

For each $t > 0$, define the integer $k_0 = k_0(t)$ by requiring $4^k t < 4^{k+1}$. For each integer $k > k_0(t)$ define the interval $I_k(t) = (4^k - t, 4^k]$ and let $I_{k_0}(t) = [0, 2 \cdot 4^k - t]$. Define the operator $P(t)$ on $H$ for each $t \geq 0$ by $P(0) = 0$ and

$$P(t)f(x) = f(2 \cdot 4^k - t - x) \quad \text{when } x \in I_k(t), \quad k \geq k_0(t),$$

$$= 0 \quad \text{otherwise.}$$
By inspection $P(t)$ is well defined and $\|P(t)\| \leq 1$ for all $t \geq 0$.

Let $H_2 = H \oplus H$ with the usual direct sum inner product: $\langle (f, g), (h, k) \rangle = \langle f, h \rangle + \langle g, k \rangle$. Define the operators $S(t), t \geq 0$, on $H_2$ by

$$S(t) = \begin{pmatrix} V^*(t) & P(t) \\ 0 & V(t) \end{pmatrix},$$

i.e., $S(t)(f, g) = (V^*(t)f + P(t)g, V(t)g)$.

Since $\|V(t)\| = \|V^*(t)\| = 1 \geq \|P(t)\|$ for all $t \geq 0$ it follows that $\|S(t)\| \leq 2$ for all $t \geq 0$. We now establish that $S(0) = I$ and that $S(t)$ has the semigroup property $S(u + t) = S(u)S(t)$ for all $u, t > 0$. Since $P(0) = 0$, we have $S(0) = I$. The semigroup property translates into the following equality:

$$(V^*(u)P(t)g(x) + P(u)V(t)g(x)) = P(u + t)g(x)$$

for all $g$ in $H$ and $u, t \geq 0$, where the fact that $V(t)$ and $V^*(t)$ are semigroups has been used. Rewriting (1) with the definitions of $V(t)$, $V^*(t)$, and $P(t)$, we must prove that $h_1(x) + h_2(x) = h_3(x)$ where

$$h_1(x) = g(2 \cdot 4^k - t - x - u) \quad \text{when } x + u \in I_k(t), \quad k \geq k_0(t),$$
$$= 0 \quad \text{otherwise};$$

$$h_2(x) = g(2 \cdot 4^k - t - x - u) \quad \text{when } x \in I_k(u), \quad x \leq 2 \cdot 4^k - t - u, \quad k \geq k_0(u),$$
$$= 0 \quad \text{otherwise};$$

$$h_3(x) = g(2 \cdot 4^k - t - x - u) \quad \text{when } x \in I_k(u + t), \quad k \geq k_0(u + t),$$
$$= 0 \quad \text{otherwise}.$$

Letting $g, u, t, x$ be fixed but arbitrary, and noting that the arguments of $h_1, h_2,$ and $h_3$ are equal, we need only verify that the conditions imposed on $x$ for these functions enable us to obtain the desired equality. Since the computations are not entirely obvious we include some of them, splitting the proof into three parts.

(1) If $h_1(x) \neq 0$, then $h_1(x) = h_3(x)$. Indeed, if $h_1(x) \neq 0$, then $x + u \in I_k(t), \quad k \geq k_0(t)$. First suppose $k > k_0(t)$ so that $t \leq 4^k$ and $x \in (4^k - t - u, 4^k - u]$ (and $x \geq 0$). If $4^k - t - u \geq 0$, then $u + t \leq 4^k$ and $k > k_0(u + t)$ so that $x \in I_k(u + t)$ and $h_3(x) = h_3(x)$. If $4^k - t - u < 0$, then $4^k - u + t \leq 4^k + 1$ (since $u \leq 4^k$ and $t \leq 4^k$) and $k_0(u + t) = k$. In this case, $4^k - t \geq 0$ implies $2 \cdot 4^k - t - u \leq 4^k - u$ so that $x \in I_{k_0}(u + t)$ and again $h_1(x) = h_3(x)$. Finally, suppose $k = k_0(t)$. Then $x \in [0, 4^k - u - t]$ implies $k = k_0(u + t)$ and $x \in I_{k_0}(u + t)$, so $h_1(x) = h_3(x)$. Similar arguments show that if $h_2(x) \neq 0$, then $h_2(x) = h_3(x)$.

(2) Either $h_1(x)$ or $h_2(x)$ must be zero (for each fixed $x, t,$ and $u$). For if both were nonzero we would have $x + u \in I_k(t)$ for some $k \geq k_0(t)$.
and \(x \in I_j(u)\) for some \(j \geq k_0(u)\). If \(k > k_0(t)\) and \(j > k_0(u)\) then \((4^k - u - t, 4^k - u]\) and \((4^j - u, 4^j]\) clearly have empty intersection, yielding a contradiction. Likewise, if \(k > k_0(t)\) and \(j = j_0(u)\), we have \(4^j < u \leq 4^{j+1}\) and \(u + t \leq 2 \cdot 4^j\). Hence \(t < 4^j\), from which it follows that \(k < j\) and \(2 \cdot 4^j > 4^k\) and thus \(2 \cdot 4^j - u - t > 4^k - u - t\). Once again \(x + u \in I_k(t)\) and \(x \in I_j(u)\) are incompatible. The remaining two cases in which \(k = k_0(t), j > k_0(u)\) and \(k = k_0(t), j = k_0(u)\) follow similarly and we conclude that either \(h_1(x)\) or \(h_2(x)\) must be zero.

(3) If \(h_3(x) \neq 0\) then either \(h_1(x)\) or \(h_2(x)\) is nonzero. Here we have \(x \in I_k(u + t),\ k \geq k_0(u + t)\). If \(k > k_0(u + t)\), then \(k > k_0(u)\) and \(k > k_0(t)\) so that \(I_k(u + t) = I_k(u) \cup [x : x + u \in I_k(t)]\). Thus one of \(h_1(x)\) and \(h_2(x)\) must be equal to \(h_2(x)\) and hence nonzero. Finally if \(k = k_0(u + t)\), then \(x \in I_k(u + t)\). If either \(k_0(t) = k_0(u + t)\) or \(k_0(u) = k_0(u + t)\) then \(x \in I_k(t)\) or \(x \in I_k(u)\) and \(h_1(x)\) or \(h_2(x)\) is nonzero. If both \(k_0(t) < k_0(u + t)\) and \(k_0(u) < k_0(u + t)\), then \(I_k(u + t) \subseteq I_k(u) \cup [x : x + u \in I_k(t)]\) and again one of the \(h_1(x)\) and \(h_2(x)\) must be nonzero.

Taking (1), (2), and (3) together we obtain \(h_1(x) + h_2(x) = h_3(x)\) so that \(S(t)\) is a semigroup.

To show that \(S(t)\) is a \((C_0)\) semigroup we must show that for each \((f, g)\) in \(H_2\), \(S(t)(f, g)\) is continuous in \(t\) on \([0, \infty)\). By the semigroup property this reduces to showing that \(\|S(t)(f, g) - (f, g)\| \to 0\) as \(t \to 0\). To this end we have

\[
\begin{align*}
\|S(t)(f, g) - (f, g)\|^2 &= \|(V^*(t)f + P(t)g - f, V(t)g - g)\|^2 \\
&= \|V^*(t)f + P(t)g - f\|^2 + \|V(t)g - g\|^2 \\
&\leq 2\|V^*(t)f - f\|^2 + 2\|P(t)g\|^2 + \|V(t)g - g\|^2.
\end{align*}
\]

The first and third terms of the last line get small as \(t \to 0\) by the continuity of translation on \(L_2([0, \infty))\) (or equivalently since \(V(t)\) and \(V^*(t)\) are \((C_0)\) semigroups). Also \(\|P(t)g\| \to 0\) because \(\|P(t)g\| = \|X(t)g\|\) where \(X(t)\) is multiplication by the characteristic function of \(U^*_{t=t_0} I_k(t)\), and \(\|X(t)g\| \to 0\) by the Lebesgue dominated convergence theorem.

Thus \(S(t)\) is a uniformly bounded \((\|S(t)\| \leq 2 \text{ for all } t \geq 0)\) \((C_0)\) semigroup and by the general theory of \((C_0)\) semigroups, \(S(t)\) has an infinitesimal generator \(L\) defined for \(y \in H_2\) by

\[
Ly = \lim_{t \to 0} \frac{S(t)y - y}{t} \quad \text{whenever this limit exists.}
\]

**Theorem.** There exists a uniformly bounded \((C_0)\) semigroup \(S(t)\) whose generator \(L\) is not similar to a dissipative operator.

**Proof.** In view of Lemma 1 it suffices to show that
Setting \( g = \text{characteristic function of } (0, 1) \) and defining \( R = \{2 \cdot 4^k - 1 : k = 0, 1, 2, \cdots \} \), the definition of \( P(t) \) shows that \( P(r)g(x) = g(x) \) for all \( r \in R \). Indeed \( I_{h_i}(r) = [0, 1] \) for such \( r \). Now using the fact that \( \langle V(r)g, h \rangle \to 0 \) as \( r \to \infty \) for all \( h \in H \),

weak limit \( (r \to \infty, r \in R)S(r)(0, g) = \) weak limit \( (r \to \infty, r \in R)(g, V(r)g) = (g, 0) \).

Also for \( (h, k) \in W(L^*) \),

\[
\langle (g, 0), (h, k) \rangle = \lim (r \to \infty, r \in R)\langle S(r)(0, g), (h, k) \rangle = \lim (r \to \infty, r \in R)\langle (0, g), S^*(r)(h, k) \rangle = 0
\]

by the definition of \( (h, k) \in W(L^*) \). Thus \( (g, 0) \in \left[ W(L^*) \right] \perp \) and clearly \( (g, 0) \in W(L) \), so \( (g, 0) \in W(L) \cap \left[ W(L^*) \right] \perp \) and \( L \) is not similar to a dissipative operator.

An investigation into the nature of the generator \( L \) of \( S(t) \) yields the following interesting result. The domain of \( L \) consists of all \( (f, g) \in H_2 \) for which

1. \( g \) is (modulo null functions) absolutely continuous on \( H \), \( \lim_{x \to 0} g(x) = 0 \), and \( dg/dx \in H \).

2. \( f \) is absolutely continuous everywhere except possibly at the points \( 4^k, k \) integral.

3. Letting \( h(x) = f(x) + \sum_{4^k < x} g(4^k) \), then \( dh/dx \in H \), so that the jumps of \( f \) at \( 4^k \) are the negatives of the values \( g(4^k) \).

For \( (f, g) \in \text{Domain } (L) \), \( L(f, g) = (dh/dx, -dg/dx) \).

In conclusion, the counterexample presented is a \( (C_0) \) semigroup with unbounded generator. This leaves open the question of whether or not the (bounded) generator of a uniformly bounded, uniformly continuous semigroup must be similar to a dissipative operator.

References