THE ARF INVARIANT FOR KNOT TYPES

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The purpose of this paper is to prove Theorem 1 below which gives a simple relation between the Arf invariant $\phi(k)$ and the Minkowski unit $C_2(k)$, $p = 2$, of a tame knot $k$ in 3-space.

**Theorem 1.** $C_2(k) = (-1)^\phi(k)$.

The cobordism invariance of $\phi(k)$ which is proved by Robertello [3] follows from this theorem and Corollary 3.5 in [2]. Further, if we denote by $\Delta(t)$ the Alexander polynomial of $k$, then a simple calculation leads to the following

**Theorem 2.** $\phi(k) = 0$ iff $\Delta(-1) \equiv \pm 1 \pmod{8}$.

1. **Seifert matrix.** Let $k$ be an oriented tame knot in 3-space and let $S$ be a Seifert surface of $k$. $S$ is a 2-cell with $2h$ bands $B_1, \ldots, B_{2h}$, where $h$ is the genus of $S$. Let $V_{2h} = (v_{ij})$ be the Seifert matrix associated to $S$. $V_{2h}$ is an integral $2h \times 2h$ matrix and the $(i, j)$ entry of the symmetric matrix $M = V_{2h} + V_{2h}$ is odd iff $(i, j) = (2r - 1, 2r)$ or $(2r, 2r - 1)$, and hence $\det M$ is odd.

Let $V_m (m \leq 2h)$ be the principal minor consisting of the first $m$ rows and columns of $V_{2h}$. Then $V_{2h}$ can be considered as the Seifert matrix associated to the surface $S'$ obtained from $S$ by removing $2h - 2l$ bands $B_{2l+1}, \ldots, B_{2h}$. Let $D_m = \det (V_m + V_m')$.

**Lemma 1.** Let $1 \leq n \leq h$ and $D_0 = 1$. Then $D_{2n-2}D_{2n} \equiv -1$ or $3 \pmod{8}$ according as $v_{2n-1,2n-1}$ $v_{2n,2n}$ is even or odd. Moreover, $D_{2n-1}$ is even, but if $v_{2n-1,2n-1}$ is odd then $D_{2n-1} \equiv 2 \pmod{4}$.

**Proof.** Let $M = V_{2n} + V_{2n}'$. We know\(^1\) that

$$D_{2n-2}D_{2n} = D_{2n-1} \det M \left(2n - 1\right) - \left\{ \det M \left(\begin{array}{c} 2n - 1 \\ 2n \end{array}\right) \right\}^2.$$  

(For a proof, for example, see [1, p. 7].) Since

$$\det M \left(\begin{array}{c} 2n - 1 \\ 2n \end{array}\right)$$

is odd,

\(^1\) $M(\xi)$ denotes the matrix obtained from $M$ by deleting the $p$th row and $q$th column, and $M(\xi) = M(\xi)$.  

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Hence, \( D_{2n-2} D_{2n} = D_{2n-1} \) \( \equiv D_{2n-1} \) (mod 8). Further, \( \det \tilde{M}(2n-1) \equiv 2v_{2n-1,2n} D_{2n-2} \) (mod 4), \( D_{2n-1} \equiv 2v_{2n-1,2n-1} D_{2n-2} \) (mod 4) and \( D_{2n-2} \) is odd. Therefore, \( \det \tilde{M}(2n-1) \equiv 2v_{2n-1,2n-1} D_{2n-2} \) (mod 8) and \( D_{2n-1} \equiv 2v_{2n-1,2n-1} \) (mod 4). This proves Lemma 1.

Since \( D_{2n-2} D_{2n} \equiv -1 \) (mod 4) by Lemma 1, it follows

**Lemma 2.** \( D_{2n} \equiv (-1)^n \) (mod 4) for \( 0 \leq n \leq h \).

Let \((a, b)_2\) denote the Hilbert symbol. Then Lemma 2 implies

**Lemma 3.** \( (-1, D_{2n})_2 = (-1)^n \).

Further, we can prove

**Lemma 4.** \((D_{2n-1}, -D_{2n-3} D_{2n})_2 = 1 \) or \(-1\) according as \( v_{2n-1,2n-3} v_{2n,2n} \) is even or odd.

**Proof.** Let us write \( D_{2n-1} = 2^m t \), where \( m \) is a nonnegative integer, \( t \) is odd and let \( q = D_{2n-3} D_{2n} \). Then \((D_{2n-1}, -q)_2 = (2, -q)_2^m(t, -q)_2 \). If \( v_{2n-1,2n-3} v_{2n,2n} \) is even, then \( q \equiv -1 \) (mod 8) by Lemma 1. Hence \((2, -q)_2 = 1 \) and \((t, -q)_2 = 1 \). Thus \((D_{2n-1}, -q)_2 = 1 \). If \( v_{2n-1,2n-3} v_{2n,2n} \) is odd, then \( D_{2n-1} \) is not divisible by 4, i.e. \( m = 1 \), and \( q \equiv 3 \) (mod 8). Therefore, \((D_{2n-1}, -q)_2 = (2, -q)_2(t, -q)_2 = -1 \).

2. **Proof of Theorem 1.** Given a Seifert matrix \( V_{2n} \) of a knot \( k \), we can define the Arf invariant \( \phi(k) \) [3] and the Minkowski unit \( C_2(k) \) [2] as follows.

\[
\phi(k) = \sum_{i=1}^{h} v_{2i-1,2i-1} v_{2i,2i} \quad \text{mod} \ 2,
\]

and

\[
C_2(k) = (-1)^\beta (-1, -D_{2h})_2 \prod_{i=1}^{2h-1} (D_i, -D_{i+1})_2,
\]

where \( \beta = [h/2] + (1 + h)(D_{2h} + 1)/2 \).

Since \( D_{2h} \equiv (-1)^h \) (mod 4) by Lemma 2, we see that \( \beta \equiv [h/2] + h + 1 \) (mod 2).

Now Lemma 2 shows that \( D_{2n} \) is not zero, while \( D_{2n-1} \) may be zero. If \( D_{2n-1} \) is zero, \((D_{2n-3}, -D_{2n-1})_2 \) and \((D_{2n-1}, -D_{2n})_2 \) are interpreted to be \((D_{2n-3}, -1)_2 \) and \((1, -D_{2n})_2 \), respectively.

Now the proof of Theorem 1 will proceed by induction on \( h \), the genus of \( S \).
For $h = 0$, the theorem is obvious. Suppose $h = 1$. Since $\beta = 1$ and $(-1, -D_2)_2 = 1$ by Lemma 3, it suffices to show that

$$(D_1, -D_2)_2 = (-1)^{v_{11}v_{22}}.$$ (2.3)

If $D_1 \neq 0$, then (2.3) follows from Lemma 4. If $D_1 = 0$, that is, $v_{11} = 0$, we have to show that $C_2(k) = 1$. However, since $C_2(k)$ depends only on the $R$-equivalent class [2], to calculate $C_2(k)$ we may use $RMR' = (a_{ij})$ instead of $M = V + V'$ for some integral unimodular matrix $R$. $R$ may be chosen so that $a_{11} \neq 0$ and $a_{22} = 0$. Then $C_2(k) = (a_{11}, a_{22})_2 = 1$.

Now we suppose that the theorem is true for $n < h$, and proceed to prove it for $n = h$.

Consider the surface $S'$ obtained from $S$ by removing the last two bands $B_{2h-1}$ and $B_{2h}$. The genus of $S'$ is $h - 1$ and the boundary of $S'$ is a new knot $k'$. Moreover, $S'$ is a Seifert surface of $k'$ and $V_{2h-2}$ is the Seifert matrix associated to $S'$. Thus, by the induction assumption, we see that in order to prove the theorem it is sufficient to show that

$$(-1)^{\beta + \beta'}(-1, -D_{2h})_2(D_{2h-2}, -D_{2h-1})_2(D_{2h-1}, -D_{2h})_2(-1, -D_{2h-2})_2$$ (2.4)

$$= (-1)^{v_{2h-1, 2h-2}v_{2h-1, 2h}}.$$ 

where $\beta' = [(h - 1)/2] + h$.

Since $\beta + \beta' \equiv h \pmod{2}$ and

$$(-1, -D_{2h})_2(-1, -D_{2h-2})_2 = (-1)^{h+1}(-1)^{h} = -1$$

by Lemma 3, (2.4) reduces to

$$(-1)^{h+1}(-1, -D_{2h-2})_2(-1, -D_{2h-1})_2(D_{2h-2}, -D_{2h-1})_2(D_{2h-1}, -D_{2h})_2 = (-1)^{v_{2h-1, 2h-2}v_{2h-1, 2h}}.$$ (2.5)

If $D_{2h-1} = 0$, then $v_{2h-1, 2h-2}v_{2h-1, 2h}$ must, by Lemma 1, be an even number. The left-hand side of (2.5) is, by our convention,

$$(-1)^{h+1}(-1, -D_{2h-2})_2 = (-1)^{h+1}(-1)^{h-1} = 1.$$ (2.6)

Suppose $D_{2h-1} \neq 0$. Then, since

$$(D_{2h-2}, -D_{2h-1})_2 = (D_{2h-2}, -1)_2(D_{2h-2}, D_{2h-1})_2
= (-1)^{h-1}(D_{2h-2}, D_{2h-1})_2,$$

(2.5) reduces to

$$(D_{2h-3}, -D_{2h-2} D_{2h})_2 = (-1)^{v_{2h-1, 2h-2}v_{2h-1, 2h}}.$$ This follows from Lemma 4, however. Thus the proof of Theorem 1 is complete.
Since Theorem 2 can be proved easily by induction on $h$, the details will be omitted.

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