THE ARF INVARIANT FOR KNOT TYPES

KUNIO MURASUGI

The purpose of this paper is to prove Theorem 1 below which gives a simple relation between the Arf invariant \( \phi(k) \) and the Minkowski unit \( C_2(k) \), \( p = 2 \), of a tame knot \( k \) in 3-space.

**THEOREM 1.** \( C_2(k) = (-1)^{\phi(k)} \).

The cobordism invariance of \( \phi(k) \) which is proved by Robertello [3] follows from this theorem and Corollary 3.5 in [2]. Further, if we denote by \( \Delta(t) \) the Alexander polynomial of \( k \), then a simple calculation leads to the following

**THEOREM 2.** \( \phi(k) = 0 \) iff \( \Delta(-1) \equiv \pm 1 \ (\text{mod } 8) \).

1. **Seifert matrix.** Let \( k \) be an oriented tame knot in 3-space and let \( S \) be a Seifert surface of \( k \). \( S \) is a 2-cell with \( 2h \) bands \( B_1, \ldots, B_{2h} \), where \( h \) is the genus of \( S \). Let \( V_{2h} = (v_{ij}) \) be the Seifert matrix associated to \( S \). \( V_{2h} \) is an integral \( 2h \times 2h \) matrix and the \((i, j)\) entry of the symmetric matrix \( M = V_{2h} + V_{2h}^T \) is odd iff \((i, j) = (2r - 1, 2r) \) or \((2r, 2r - 1)\), and hence \( \det M \) is odd.

Let \( V_m \ (m \leq 2h) \) be the principal minor consisting of the first \( m \) rows and columns of \( V_{2h} \). Then \( V_m \) can be considered as the Seifert matrix associated to the surface \( S' \) obtained from \( S \) by removing \( 2h - 2l \) bands \( B_{2l+1}, \ldots, B_{2h} \). Let \( D_m = \det (V_m + V_m^T) \).

**LEMMA 1.** Let \( 1 \leq n \leq h \) and \( D_0 = 1 \). Then \( D_{2n-2} D_{2n} \equiv -1 \) or \( 3 \ (\text{mod } 8) \) according as \( v_{2n-1,2n-1} \) \( v_{2n,2n} \) is even or odd. Moreover, \( D_{2n-1} \) is even, but if \( v_{2n-1,2n-1} \) is odd then \( D_{2n-1} \equiv 2 \ (\text{mod } 4) \).

**PROOF.** Let \( M = V_{2n} + V_{2n}^T \). We know\(^1\) that

\[
D_{2n-2} D_{2n} = D_{2n-1} \det \tilde{M} (2n - 1) - \left\{ \det \tilde{M} \left( \begin{array}{c} 2n - 1 \\ 2n \end{array} \right) \right\}^2.
\]

(For a proof, for example, see [1, p. 7].) Since

\[
\det \tilde{M} \left( \begin{array}{c} 2n - 1 \\ 2n \end{array} \right)
\]

is odd,

Received by the editors March 8, 1968.

\(^1\) \( \tilde{M}(\xi) \) denotes the matrix obtained from \( M \) by deleting the \( p \)th row and \( q \)th column, and \( \tilde{M}(\xi) = \tilde{M}(\xi) \).

69
\[
\left\{ \det \bar{M} \left( \begin{array}{c}
2n - 1 \\
2n
\end{array} \right) \right\}^2 = 1 \pmod{8}.
\]

Hence, \( D_{2n-2} D_{2n} - D_{2n-1} = \det \bar{M} (2n-1) - 1 \pmod{8} \). Further, \( \det \bar{M} (2n-1) = 2v_{2n-2} D_{2n-2} \pmod{4}, D_{2n-1} = 2v_{2n-1} D_{2n-1} \pmod{4} \) and \( D_{2n-2} \) is odd. Therefore, \( \det \bar{M} (2n-1) D_{2n-1} = 4v_{2n-2} D_{2n-2} D_{2n-1} \pmod{8} \) and \( D_{2n-1} = 2v_{2n-1} D_{2n-1} \pmod{4} \). This proves Lemma 1.

Since \( D_{2n-2} D_{2n} = -1 \pmod{4} \) by Lemma 1, it follows

**Lemma 2.** \( D_{2n} = (-1)^n \pmod{4} \) for \( 0 \leq n \leq h \).

Let \((a, b)_2\) denote the Hilbert symbol. Then Lemma 2 implies

**Lemma 3.** \((-1, D_{2n})_2 = (-1)^n\).

Further, we can prove

**Lemma 4.** \((D_{2n-1}, -D_{2n-2} D_{2n})_2 = 1 \text{ or } -1 \) according as \( v_{2n-1}, 2n-1, v_{2n}, 2n \) is even or odd.

**Proof.** Let us write \( D_{2n-1} = 2^m t \), where \( m \) is a nonnegative integer, \( t \) is odd and let \( q = D_{2n-2} D_{2n} \). Then \((D_{2n-1}, -q)_2 = (2, -q)_2(t, -q)_2\). If \( v_{2n-1}, 2n-1, v_{2n}, 2n \) is even, then \( q \equiv -1 \pmod{8} \) by Lemma 1. Hence \((2, -q)_2 = 1 \) and \((t, -q)_2 = 1\). Thus \((D_{2n-1}, -q)_2 = 1\). If \( v_{2n-1}, 2n-1, v_{2n}, 2n \) is odd, then \( D_{2n-1} \) is not divisible by 4, i.e. \( m = 1 \), and \( q \equiv 3 \pmod{8} \). Therefore, \((D_{2n-1}, -q)_2 = (2, -q)_2(t, -q)_2 = -1\).

2. **Proof of Theorem 1.** Given a Seifert matrix \( V_{2n} \) of a knot \( k \), we can define the Arf invariant \( \phi(k) \) [3] and the Minkowski unit \( C_2(k) \) [2] as follows.

\[
\phi(k) = \sum_{i=1}^{h} v_{2i-1, 2i-1} v_{2i, 2i} \pmod{2},
\]

and

\[
C_2(k) = (-1)^h (-1, -D_{2k})_2 \prod_{i=1}^{2h-1} (D_i, -D_{i+1})_2,
\]

where \( \beta = [h/2] + (1 + h)(D_{2k} + 1)/2 \).

Since \( D_{2k} \equiv (-1)^h \pmod{4} \) by Lemma 2, we see that \( \beta \equiv [h/2] + h + 1 \pmod{2} \).

Now Lemma 2 shows that \( D_{2n} \) is not zero, while \( D_{2n-1} \) may be zero. If \( D_{2n-1} \) is zero, \((D_{2n-3}, -D_{2n-1})_2 \) and \((D_{2n-1}, -D_{2n})_2 \) are interpreted to be \((D_{2n-3}, -1)_2 \) and \((1, -D_{2n})_2 \), respectively.

Now the proof of Theorem 1 will proceed by induction on \( h \), the genus of \( S \).
For $h = 0$, the theorem is obvious. Suppose $h = 1$. Since $\beta = 1$ and $(-1, -D_2) = 1$ by Lemma 3, it suffices to show that

$$(2.3) \quad (D_1, -D_2)_2 = (-1)^{v_{11}v_{22}}.$$

If $D_1 \neq 0$, then (2.3) follows from Lemma 4. If $D_1 = 0$, that is, $v_{11} = 0$, we have to show that $C_2(k) = 1$. However, since $C_2(k)$ depends only on the $R$-equivalent class $[2]$, to calculate $C_2(k)$ we may use $RMR' = (a_{ij})$ instead of $M = V_2 + V'_2$ for some integral unimodular matrix $R$. $R$ may be chosen so that $a_{11} \neq 0$ and $a_{22} = 0$. Then $C_2(k) = (a_{11}, a_{22})_2 = 1$.

Now we suppose that the theorem is true for $n < h$, and proceed to prove it for $n = h$.

Consider the surface $S'$ obtained from $S$ by removing the last two bands $B_{2h-1}$ and $B_{2h}$. The genus of $S'$ is $h - 1$ and the boundary of $S'$ is a new knot $k'$. Moreover, $S'$ is a Seifert surface of $k'$ and $V_{2h-2}$ is the Seifert matrix associated to $S'$. Thus, by the induction assumption, we see that in order to prove the theorem it is sufficient to show that

$$(2.4) \quad (-1)^{\beta + \beta'}(-1, -D_{2h})_2(D_{2h-2}, -D_{2h-1})_2(D_{2h-1}, -D_{2h})_2(-1, -D_{2h-2})_2 = (-1)^{\beta + \beta'}(-1, -D_{2h})_2,$$

where $\beta' = [(h - 1)/2] + h$.

Since $\beta + \beta' = h \pmod{2}$ and

$$(-1, -D_{2h})_2(-1, -D_{2h-2})_2 = (-1)^{h+1}(-1)^h = -1$$

by Lemma 3, (2.4) reduces to

$$(2.5) \quad (-1)^{\beta + \beta'}(-D_{2h-2}, -D_{2h-1})_2(D_{2h-1}, -D_{2h})_2 = (-1)^{\beta + \beta'}(-1, -D_{2h})_2.$$

If $D_{2h-1} = 0$, then $v_{2h-1, 2h} = v_{2h, 2h}$ must, by Lemma 1, be an even number. The left-hand side of (2.5) is, by our convention,

$$(-1)^{\beta + \beta'}(-1, -D_{2h})_2(-1) = (-1)^{h+1}(-1)^{h-1} = 1.$$

Suppose $D_{2h-1} \neq 0$. Then, since

$$(D_{2h-2}, -D_{2h-1})_2 = (D_{2h-2}, -1)_2(D_{2h-2}, D_{2h-1})_2 = (-1)^{h-1}(D_{2h-2}, D_{2h-1})_2,$$

(2.5) reduces to

$$(2.6) \quad (D_{2h-3}, -D_{2h-2})_2 = (-1)^{2h-1, 2h}.$$

This follows from Lemma 4, however. Thus the proof of Theorem 1 is complete.
Since Theorem 2 can be proved easily by induction on $h$, the details will be omitted.

REFERENCES


