FACTORIZATION OF CERTAIN MAPS UP TO HOMOTOPY

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If \( f: X \rightarrow Y \) is a map of a space \( X \) into a space \( Y \), we say that \( f \) is a \textit{local connection in dimension} \( n \), provided that for every point \( y \in Y \) and every neighborhood \( N \) of \( y \) there is a neighborhood \( V \subset N \) of \( y \) such that for \( 0 \leq k \leq n-1 \) any map \( g: S^k \rightarrow f^{-1}V \) extends to a map \( g': B^{k+1} \rightarrow f^{-1}N \) and for any map \( g: S^n \rightarrow f^{-1}V \) the map \( fg: S^n \rightarrow V \) extends to a map \( h: B^{n+1} \rightarrow N \). Using star-refinements of open covers and a standard approximation technique we establish the following theorem (a slightly weaker form of which has been announced by Price [2]).

\textbf{Theorem 1.} Let \( Y \) be a metric space, and let \( f: X \rightarrow Y \) be a local connection in dimension \( n \) with dense image. Let \( L \) be a subcomplex of a finite simplicial complex \( K \) such that \( \dim (K-L) \leq n \), and let \( g: L \rightarrow X \) and \( h: K \rightarrow Y \) be maps such that \( h|L = fg \). Then there is a map \( g': K \rightarrow X \) such that \( g'|L = g \) and \( fg' \) is homotopic to \( h \) relative to \( L \). If \( d \) is any metric for \( Y \) and \( \epsilon > 0 \), the map \( g' \) and the homotopy \( H \) may be chosen so that for all points \( p \in K \) the diameter (with respect to \( d \)) of \( H(p \times I) \) is <\( \epsilon \).

This implies that \( f \) is an \( n \)-equivalence; i.e., \( f \) maps the set of path-components of \( X \) bijectively to the set of path-components of \( Y \), and that for every \( x \in X \), \( f|\pi_k(X, x) \subset \pi_k(Y, f(x)) \) is an isomorphism for \( 1 \leq k \leq n-1 \) and an epimorphism for \( k = n \). Since \( f|f^{-1}W \) is also a local connection in dimension \( n \) for every open set \( W \subset Y \), it follows that \( Y \) is \( LC^n \). Using these facts and the lemmas for the proof of Theorem 1 we obtain sharper forms of known results:

\textbf{Theorem 2 (cf. Smale [3]).} Let \( X \) be a paracompact \( LC^n \) space, let \( Y \) be a metric space, and let \( f: X \rightarrow Y \) be a closed map of \( X \) onto \( Y \) such that \( f^{-1}(y) \) is \( LC^{n-1} \) and \((n-1)\)-connected for every \( y \in Y \). Then \( Y \) is \( LC^n \), and \( f \) is an \( n \)-equivalence.

\textbf{Theorem 3 (cf. Kwun [1]).} Let \( M \) be a manifold, and let \( G \) be an upper semicontinuous decomposition of \( M \) into cellular sets. If the

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decomposition space $M/G$ is finite dimensional, it is a homotopy manifold.

1. All complexes will be finite simplicial complexes, and $(K, L)$ will be called an $n$-pair provided that $L$ is a subcomplex of $K$ and $\dim (K - L) \leq n$. The $q$-skeleton of $K$ will be denoted $K^q$. If $\mathcal{U}$ is a collection of open sets in $Y$, then a map $h: K \to Y$ (a homotopy $H: K \times I \to Y$) will be said to map $K$ (resp. $K \times I$) into $\mathcal{U}$ provided that for every (closed) simplex $\alpha$ of $K$ there is $U \in \mathcal{U}$ with $h(\alpha) \subset U$ (resp. $H(\alpha \times I) \subset U$). Associated with $\mathcal{U}$ is the collection $\mathcal{U}^* = \{ U^* | U \in \mathcal{U} \}$, where $U^* = \bigcup \{ U' \in \mathcal{U} | U \cap U' \neq \emptyset \}$. A map $f: X \to Y$ will be said to be a strong local connection in dimension $n$, if for every point $y \in Y$ and every neighborhood $N$ of $y$ there is a neighborhood $V \subset N$ of $y$ such that for $0 \leq k \leq n$ any map $g: S^k \to f^{-1}V$ extends to a map $g': B^{k+1} \to f^{-1}N$.

Let $\mathcal{U}$ and $\mathcal{V}$ be open covers of a space $X$ such that $\mathcal{V}$ refines $\mathcal{U}$, and let $f: X \to Y$ be a map. For every nonnegative integer $n$ we define assertions $E(\mathcal{U}, \mathcal{V}; n)$, $H(\mathcal{U}, \mathcal{V}; n)$, and $H(\mathcal{U}, \mathcal{V}; f; n)$ as follows:

$E(\mathcal{U}, \mathcal{V}; n)$. If $(K, L)$ is any $n$-pair and $g: L \to X$, $h: K \to Y$ are any maps such that $h$ extends $fg$ and maps $cl(K - L)$ into the collection $\{ V \in \mathcal{V} | f(X) \cap V \neq \emptyset \}$, then there is an extension $g': K \to X$ of $g$ such that for every simplex $\alpha$ of $cl(K - L)$ there is $U \in \mathcal{U}$ with $fg'(\alpha) \subset h(\alpha) \subset U$.

$H(\mathcal{U}, \mathcal{V}; n)$. If $(K, L)$ is any $n$-pair and $g: L \to X$, $g': K \to X$, $g'': K \to X$ are any maps such that $g = g'|L = g''|L$ and for every simplex $\alpha$ of $K$ there is $V \in \mathcal{V}$ with $g'(\alpha) \subset g''(\alpha) \subset f^{-1}V$, then there is a homotopy $G: g' \simeq g''$ relative to $L$ which maps $K \times I$ into $f^{-1}\mathcal{U} = \{ f^{-1}U | U \in \mathcal{U} \}$.

$H(\mathcal{U}, \mathcal{V}; f; n)$. If $(K, L)$ is any $n$-pair and $g: L \to X$, $g': K \to X$, $g'': K \to X$ are any maps such that $g = g'|L = g''|L$ and for every simplex $\alpha$ of $K$ there is $V \in \mathcal{V}$ with $g'(\alpha) \subset g''(\alpha) \subset f^{-1}V$, then there is a homotopy $H: fg' \simeq fg''$ relative to $L$ which maps $K \times I$ into $\mathcal{U}$.

**Lemma 1.** Let $Y$ be paracompact, and let $f: X \to Y$ be a strong local connection in dimension $n$. Then for any open cover $\mathcal{V}$ of $Y$ there is an open cover $\mathcal{U}$ of $Y$ refining $\mathcal{V}$ such that both $E(\mathcal{U}, \mathcal{V}; n+1)$ and $H(\mathcal{U}, \mathcal{V}; f; n)$ hold.

**Proof.** For $n = -1$ there are no conditions on the map $f$, and both assertions hold for $\mathcal{U} = \mathcal{U}$. Assume that the lemma is true for $n < k$, and let $f: X \to Y$ be a strong local connection in dimension $k$. If $\mathcal{U}$ is an open cover of $Y$, let $\mathcal{W}$ be an open cover such that for each $W \in \mathcal{W}$ there is $U \in \mathcal{U}$ such that $W^* \subset U$ and any map $S^k \to f^{-1}(W^*)$ extends to a map $B^{k+1} \to f^{-1}U$. 

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Let $\mathcal{V}$ be an open cover refining $\mathcal{W}$ such that both $E(\mathcal{V}, \mathcal{W}; k)$ and $H(\mathcal{V}, \mathcal{W}; k-1)$ hold. Then both $E(\mathcal{V}, \mathcal{U}; k+1)$ and $H(\mathcal{V}, \mathcal{U}; k)$ hold. In fact if $K$, $L$, $g$, $h$ are as in the first assertion, there is an extension $g''': L \cup K^k \to X$ such that for every $k$-simplex $\beta$ of $cl(K-L)$ there is $W(\beta) \subseteq \mathcal{W}$ with $f g''''(\beta) \cup h(\beta) \subseteq W(\beta)$. Let $\alpha$ be a $(k+1)$-simplex of $K-L$, and let $W \in \mathcal{W}$ be such that $h(\alpha) \subseteq W$. If $\beta < \alpha$, then $W \cap W(\beta) \neq \emptyset$. It follows that $f g''''(\partial \alpha) \subseteq W^*$; hence there is $U \in \mathcal{U}$ such that $W^* \subseteq U$, and there is an extension $\alpha \to f^{-1} U$ of $g''': \partial \alpha \to f^{-1}(W^*)$.

Combining such extensions gives the desired map $g': K \to X$; thus $E(\mathcal{V}, \mathcal{U}; k+1)$ holds. On the other hand if $K$, $L$, $g$, $g'$, $g''$ are as in the second assertion, let $G': (L \cup K^{k-1}) \times I \to X$ be a homotopy as in $H(\mathcal{V}, \mathcal{W}; k-1)$, and extend $G'$ to $G'': K \times \{0, 1\} \cup (L \cup K^{k-1}) \times I \to X$ by $G''(p, 0) = g'(p)$ and $G''(p, 1) = g''(p)$ for all points $p \in K$. For any $k$-simplex $\alpha$ of $K$ there is $W \in \mathcal{W}$ such that $G''(\alpha \times \{0, 1\}) \subseteq f^{-1}(W^*)$; thus $G''(\partial (\alpha \times I)) \subseteq f^{-1}(W^*)$. Then $G''$ extends to a homotopy $G: K \times I \to X$ which maps $K \times I$ into $\mathcal{U}$, which proves that $H(\mathcal{V}, \mathcal{U}; k)$ holds.

**Lemma 2.** Let $Y$ be paracompact, and let $f: X \to Y$ be a local connection in dimension $n$. Then for any open cover $\mathcal{U}$ of $Y$ there is an open cover $\mathcal{V}$ refining $\mathcal{U}$ such that $H(\mathcal{V}, \mathcal{U}; f; n)$ holds.

**Proof.** If $n = -1$, there is nothing to prove. Assume that the lemma is true for $n < k$, and let $f: X \to Y$ be a local connection in dimension $k$. Then $f$ is a strong local connection in dimension $k-1$. If $\mathcal{U}$ is an open cover of $Y$, let $\mathcal{W}$ be an open cover such that for each $W \in \mathcal{W}$ there is $U \in \mathcal{U}$ such that $W^* \subseteq U$ and for any map $h: S^k \to f^{-1}(W^*)$ the map $fh$ extends to a map $B^{k+1} \to U$. By Lemma 1 there is an open cover $\mathcal{V}$ of $Y$ refining $\mathcal{W}$ such that $H(\mathcal{V}, \mathcal{W}; k-1)$ holds. To see that $H(\mathcal{V}, \mathcal{U}; f; k)$ holds consider $K$, $L$, $g$, $g'$, and $g''$ as in the assertion, and let $G: (L \cup K^{k-1}) \times I \to X$ be a homotopy as in $H(\mathcal{V}, \mathcal{W}; k-1)$. Extend $G$ to a map $G': K \times \{0, 1\} \cup (L \cup K^{k-1}) \times I \to X$ by $G'(p, 0) = g'(p)$ and $G'(p, 1) = g''(p)$, and observe that for every $k$-simplex $\alpha$ of $K-L$ there is $W \in \mathcal{W}$ with $f G'(\partial (\alpha \times I)) \subseteq W^*$. It follows that $f G'$ extends to a homotopy $H$ which maps $K \times I$ into $\mathcal{U}$.

**2. Proof of Theorem 1.** Let $Y$ have metric $d$. Using Lemmas 1 and 2 choose a sequence $\{\mathcal{V}_r\} 0 \leq r < \infty$ of open covers of $Y$ such that mesh $\mathcal{V}_r < \epsilon/4(r+1)$, $\mathcal{V}_{r+1}$ refines $\mathcal{V}_r$, $E(\mathcal{V}_{r+1}, \mathcal{V}_r; n)$ holds, and $H(\mathcal{V}_{r+1}, \mathcal{V}_r; f; n)$ holds. (For a given $\epsilon$ such a sequence provides the extension and the homotopy in all cases.)

If $K$, $L$, $g$, and $h$ are as in Theorem 1, choose a sequence $\{K_r\} 1 \leq r < \infty$ of subdivisions of $K$ such that $K_{r+1}$ is a subdivision
of $K_r$ and $h$ maps $K_r$ into $\mathcal{U}_{r+1}$. Using the fact that $E(\mathcal{U}_{r+1}, \mathcal{U}_r; n)$ holds, choose extensions $g_r: K_r \rightarrow X$ of $g$ ($1 \leq r < \infty$) such that for every $\alpha$ in $K_r$ there is $V \in \mathcal{U}_r$ with $f_{gr}(\alpha) \cup h(\alpha) \subseteq V$. Since mesh $\mathcal{U}_r < \varepsilon/4(r+1)$, $d(f_{gr}(p), h(p)) < \varepsilon/4(r+1)$ for all points $p \in K_r$.

Set $g' = g_l$, and construct $H$ by "filling in" between $g_r$ and $g_{r+1}$ as follows. Let $\alpha$ be a simplex of $K_r$, and let $f_{gr}(\alpha) \cup h(\alpha) \subseteq V$ for some $V \in \mathcal{U}_r$. Consider $\alpha$ as a subcomplex of $K_{r+1}$, and observe that for every simplex $\beta$ of $\alpha$, there is $V' \in \mathcal{U}_r$ such that $f_{gr+1}(\beta) \cup h(\beta) \subseteq V'$; hence $f_{gr+1}(\alpha) \cup f_{gr}(\alpha) \subseteq V'$. Since $H(\mathcal{U}_r, \mathcal{U}_{r-1}; f; n)$ holds, there is a homotopy $H_r: f_{gr+1} \simeq f_{gr}$ relative to $L$, which may be considered as a map $H_r: K \times [1/(r+1), 1/r] \rightarrow Y$, such that the diameter of $H_r(\alpha \times [1/(r+1), 1/r])$ is $< \varepsilon/4r$ for every simplex $\alpha$ of $K_r$. This implies that $d(H_r(p, t), h(p)) < \varepsilon/2r$ for all $(p, t) \in K \times [1/(r+1), 1/r]$. Define $H_r: K \times I \rightarrow Y$ by $H_r(p, t) = H_r(p, t)$ for $1/(r+1) \leq t \leq 1/r$ and by $H_r(p, 0) = h(p)$. It is easy to check that $H_r$ is a map and is in fact an $\varepsilon$-homotopy relative to $L$. This completes the proof.

3. Proof of Theorem 2. Since $f$ is a closed map and $f^{-1}(y)$ is $(n-1)$-connected for every $y \in Y$, to show that $f$ is a local connection in dimension $n$ it suffices to show that for every open neighborhood $U$ of $f^{-1}(y)$ there is an open neighborhood $V \subseteq U$ of $f^{-1}(y)$ such that for $0 \leq k \leq n$ any map $S^k \rightarrow V$ is homotopic in $U$ to a map $S^k \rightarrow f^{-1}(y)$. Let $A$ be the inverse set under $f$ of a point of $Y$. Since for any closed LC$^n-1$ subset $A$ of an LC$^n$ space $X$, the inclusion map $i: A \subseteq X$ is a local connection in dimension $n$, and since $X$ is paracompact, Lemma 1 applies to $i: A \subseteq X$ and $1: X \subseteq X$. If $U$ is an open neighborhood of $A$, let $\mathcal{U}$ be the cover of $X$ consisting of $U$ and $X - A$, and $\mathcal{W}$ be an open cover of $X$ refining $\mathcal{U}$ such that $H(\mathcal{W}, \mathcal{U}; n)$ holds for $1 \subseteq X \subseteq X$, and let $\mathcal{W}$ be an open cover refining $\mathcal{W}$ such that $E(\mathcal{U}, \mathcal{W}; n)$ holds for $i: A \subseteq X$.

Set $V = U \{ V' \in \mathcal{U} | V' \cap A \neq \emptyset \}$. For $0 \leq k \leq n$ triangulate $S^k$ in some way as a complex $K$, and observe that for any map $h: K \rightarrow V$ there is a subdivision $K'$ of $K$ such that $h$ maps $K'$ into $\{ V' \in \mathcal{U} | V' \cap A \neq \emptyset \}$. It follows that there is a map $g: K' \rightarrow A$ such that for each $\alpha$ in $K'$ there is $W \subseteq \mathcal{W}$ with $g(\alpha) \cup h(\alpha) \subseteq W$. Since $H(\mathcal{W}, \mathcal{U}; n)$ holds for $1 \subseteq X \subseteq X$, there is a homotopy $H: \mathcal{W} \times I \rightarrow X$ such that $H$ maps $K' \times I$ into $\mathcal{U}$. Since $g(K') \subseteq A$, $H(K' \times I) \subseteq U$. This completes the proof.

4. Proof of Theorem 3. We recall that $M$ is an $n$-manifold, if it is a separable metric space each point of which has an open neighborhood homeomorphic to $R^n$ and that a subset $A$ of $M$ is cellular, if $A = \bigcap_{i=1}^{\infty} Q_i$, where $Q_i$ is a closed $n$-cell ($1 \leq j < \infty$) and $\text{int } Q_i \supseteq Q_{i+1}$.
If \( G \) is an upper semicontinuous decomposition of \( M \) into cellular subsets, then it is well known that \( M/G \) is a separable metric space and that the projection \( P: M \to M/G \) is a closed map. It follows directly from the definition of cellularity that \( P \) is a strong local connection in dimension \( k \) for all \( k \), and therefore that \( M/G \) is LC\( ^\infty \).

In order to prove that for every point \( x \in M/G \) and every neighborhood \( N \) of \( x \) there are (connected) open neighborhoods \( V, U \) of \( x \) such that \( V \subset U \subset N \) and for all \( k \) the image of \( \pi_k(V-x) \) in \( \pi_k(U-x) \) (under the homomorphism induced by the inclusion \( V-x \subset U-x \)) is isomorphic to \( \pi_k(S^{n-1}) \) we could duplicate the arguments of [1] using the fact that \( P|P^{-1}(W) \) is a local connection in all dimensions for every open set \( W \) of \( M/G \) wherever Smale's theorem is used. We shall omit these details.

**References**


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