A NOTE ON COVER AND AVOIDANCE PROPERTIES
IN SOLVABLE GROUPS

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In [1] Carter showed the existence within a finite solvable group of a conjugate class of nilpotent self normalizing subgroups. In [2] Gaschütz defined Carter's work in the more general setting of formation theory. In [1] Carter showed that his groups satisfy a cover and avoidance property with respect to suitable factors of G. This note will extend this result within the framework of formation theory.

All groups considered will be finite and solvable. All definitions appear in [1] or [2] and all notations will be standard save where explicit definitions are given. \( \mathbb{F} \) will be a formation whose \( \mathbb{F} \)-groups may or may not exist. \( G(\mathbb{F}) \) will be the unique normal subgroup of \( G \) minimal with respect to the property that \( G/G(\mathbb{F}) \in \mathbb{F} \).

**Definition.** If \( M=G \), a series \( \langle 1 \rangle = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G \) is called an \( M \)-series of \( G \) if \( G_i \) is normalized by \( M \) and \( G_i/G_{i-1} \) is a non-trivial irreducible \( M \)-factor.

**Theorem 1.** If \( F \) is an \( \mathbb{F} \)-group of \( G \) and \( \{ G_i \} \), \( 0 \leq i \leq n \) is any \( F \)-series of \( G \) then \( F \) covers \( G_i/G_{i-1} \) if and only if \( FG_i/G_{i-1} \in \mathbb{F} \).

**Proof.** Suppose \( FG_i/G_{i-1} \in \mathbb{F} \) and \( F \) avoids \( G_i/G_{i-1} \). Since \( (F \cap G_{i-1}) G_{i-1} = G_{i-1} \) we have \( F \cap G_i \triangleleft G_{i-1} \) and \( F \cap G_i = F \cap G_{i-1} \). On the other hand since \( F \) is an \( \mathbb{F} \)-group, \( F \triangleleft FG_i \) and \( FG_i/G_{i-1} \in \mathbb{F} \) we get \( FG_{i-1} = FG_i \). Since \( G_{i-1} \) is proper in \( G_i \) we get a contradiction to \( F \cap G_i = F \cap G_{i-1} \). Suppose \( FG_i/G_{i-1} \in \mathbb{F} \) and \( F \) covers \( G_i/G_{i-1} \). We have \( FG_i = FG_{i-1} \). Thus \( FG_i/G_{i-1} = FG_i/G_{i-1} = F/F \cap G_{i-1} \in \mathbb{F} \). This is a contradiction.

What is more interesting is that this cover avoidance property actually characterizes the \( \mathbb{F} \)-groups of \( G \).

**Theorem 2.** If \( G \) is finite solvable and \( \{ G_i \} \) is an \( M \)-series of \( G \) such that \( M \) covers \( G_i/G_{i-1} \) if and only if \( MG_i/G_{i-1} \in \mathbb{F} \) then \( M \) is an \( \mathbb{F} \)-group of \( G \).

**Proof.** The proof will be by induction on \( (G: M) \left| G \right. \). If \( (G: M) \left| G \right. \)

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not cover all the factors of the series \( \{G_t\} \). Choose \( t \) maximal so that \( M \) does not cover \( G_t/G_{t-1} \). Then it follows that \( G = MG_t \), \( G \triangleleft G \), \( G_{t-1} \triangleleft G \), \( G/G_t \triangleleft \mathfrak{F} \) and \( G/G_{t-1} \triangleleft \mathfrak{N} \). In particular \( MG_{t-1} \triangleleft G \). It is easy to see that \( \{\Lambda_t\} \) where \( \Lambda_t = G_t \cap MG_{t-1} \) forms an \( M \)-series of \( MG_{t-1} \) satisfying the hypothesis of the theorem. Also the homomorphic images of \( \{G_t\} \) show that \( MG_{t-1}/G_{t-1} \) satisfies the hypotheses of the theorem in \( G/G_{t-1} \). If \( |G_{t-1}| > 1 \) then induction on \( M \) in \( MG_{t-1} \) and \( MG_{t-1}/G_{t-1} \) in \( G/G_{t-1} \) yields that \( M \) is an \( \mathfrak{F} \)-group in \( MG_{t-1} \) and \( MG_{t-1}/G_{t-1} \) is an \( \mathfrak{N} \)-group in \( G/G_{t-1} \). By [2, Lemma 2.3], it follows that \( M \) is an \( \mathfrak{N} \)-group of \( G \). Thus \( |G_{t-1}| = 1 \) and \( G_t \) is a minimal normal subgroup of \( G \). It follows that \( M \) is maximal in \( G \) and \( M \cap G_t = \langle 1 \rangle \). Thus since \( G/G_t \in \mathfrak{F} \) and \( G \in \mathfrak{N} \) we have that \( G_t \) is precisely \( G(\mathfrak{N}) \). It follows that \( M \) is an \( \mathfrak{N} \)-group of \( G \).

To bring Carter's cover avoidance property under our theorem we prove

**Theorem 3.** Suppose \( M < G \) and \( M \) is nilpotent. Let \( H/K \) be an irreducible \( M \)-factor of \( G \). Then \( MH/K \) is nilpotent if and only if \( H/K \) is \( M \)-central.

**Proof.** If \( MH/K \) is nilpotent since \( H/K \) is minimal normal in \( MH/K \) we have that \( H/K < Z(MH/K) \). Thus \( [M, H] < K \) or \( H/K \) is \( M \)-central. If \( H/K \) is \( M \)-central then \( MH/K \) is the product of two normal nilpotent groups \( MK/K \) and \( H/K \). Thus \( MH/K \) is nilpotent.

**References**


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