ON THE ZEROS OF THE BERGMAN FUNCTION IN DOUBLY-CONNECTED DOMAINS

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The purpose of this note is to show that every doubly-connected Lu Qi-Keng domain in \( C^1 \) is pseudoconformally equivalent to a disc with the center deleted. This extends a result of M. Skwarczynski [4], who gave an example of a domain \( C^1 \) which is not a Lu Qi-Keng domain. (See definition below.) Our results indicate that, at least in this particular case, there exists a connection between the degree of connectivity of \( D \) and zeros of the Bergman function. We use the notation \( z = (z^1, z^2, \cdots, z^n) \) for a point in \( D \subseteq C^n \) and \( i \) for \((\bar{p}, \bar{p}, \cdots, \bar{p})\). We denote

\[
D^* = \{ i \mid i \in D \}.
\]

Definitions and theorems relating to the Bergman function can be found in [1]. In what follows, the Bergman function of the domain \( D \subseteq C^n \) will be denoted by \( K_D(z, i) \).

**Definition.** A domain \( D \subseteq C^n \) is a Lu Qi-Keng domain if the equation \( K_D(z, i) = 0 \) has no solution in \( D \times D^* \) (see [4]).

**Theorem 1.** Let \( D \) be the ring \( 0 < r < |z| < 1 \). Then \( D \) is not a Lu Qi-Keng domain.

**Proof.** As was shown by Zarankiewicz in [5], see also [1, p. 10],

\[
K_D(z, i) = \frac{1}{\pi i} \left[ \wp \{ \log(zi); \omega, \omega' \} + \frac{\eta}{\omega} - \frac{1}{2 \omega'} \right].
\]

\( \wp \) is the Weierstrassian \( \wp \)-function, \( \omega = \pi i, \omega' = \log r, 2\eta \) is the increment of the Weierstrassian \( \zeta \)-function related to the half-period \( \omega \). (We note that since the first half-period \( \omega = \pi i \), the value of the \( \wp \)-function does not depend on the value chosen for \( \log(zi) \).) Using the Legendre equation \( \eta \omega' - \eta' \omega = \pi i/2 \) (real(\( \omega'/i\omega \)) > 0), (1) can be written as

\[
K_D(z, i) = \frac{1}{\pi i} \left[ \wp(u; \omega, \omega') + \frac{\eta'}{\omega'} \right],
\]

\( u = \log(zi) \). The function \( e^u \) maps the period-parallelogram (points

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Re \( u = 0 \), log \( r^4 \) excluded) onto the \( q \)-ring \( 0 < r^2 < |q| < 1 \), \( q = z \). Since the doubly periodic function \( \mathcal{Q}(u) \) attains every value in the period-parallelogram exactly twice, the function

\[
\eta'/\omega' + \mathcal{Q}(u)
\]

attains every value in the \( q \)-ring except for values attained when \( u \) is in the segment \( \text{Re} \, u = 0, 0 \leq \text{Im} \, u \leq \pi \). Consider the boundary of the rectangle with vertices \( 0, \pi i, \log r + \pi i, \log r \), in the \( u \)-plane with counterclockwise orientation. On this boundary, \( \mathcal{Q} \) attains real values increasing monotonically from \( -\infty \) to \( +\infty \). The function (3) has the same property, and we conclude that the exceptional values of (3) form a closed segment \([ -\infty, \eta'/\omega' + \mathcal{Q}(\pi i) \] on the real axis. We infer the Bergman function has a zero in \( D \times D^* \) if and only if

\[
\eta'/\omega' + \mathcal{Q}(\pi i) < 0.
\]

We prove next that, for every \( 0 < r < 1 \), (4) holds. Consider the new pair of primitive half-periods, \( \tilde{\omega} = -\log r, \tilde{\omega}' = \pi i \). We then obtain

\[
\eta'/\omega' + \mathcal{Q}(\pi i) = \tilde{\eta}/\tilde{\omega} + \mathcal{Q}(\pi i).
\]

It is known that [2, p. 336],

\[
\frac{\tilde{\eta}}{\tilde{\omega}} + \mathcal{Q}(u) = -\pi^2 \left\{ \frac{1}{\omega^2} \left( z - \bar{z}^{-1} \right)^2 + \sum_{m=1}^{\infty} \frac{h^{2m}g^{-2}}{(1 - h^{2m}g^{-2})^2} \right\} + \sum_{m=1}^{\infty} \frac{h^{2m}g^2}{(1 - h^{2m}g^2)^2},
\]

\[
v = u/2\tilde{\omega}, \quad z = e^{v}i, \quad t = \tilde{\omega}'/\tilde{\omega}, \quad h = e^{v} (\text{Im} \, t > 0).
\]

Since the right-hand side of (6) for \( \tilde{\omega} = -\log r, \tilde{\omega}' = \pi i \), and \( u = \pi i \) is negative for all \( 0 < r < 1 \), (4) holds. This completes the proof of Theorem 1.

**Theorem 2.** Every doubly-connected Lu Qi-Keng domain in \( C^1 \) is pseudoconformally equivalent to a disc with the center deleted.

**Proof.** Let \( D \) be a doubly-connected Lu Qi-Keng domain in \( C^1 \). It must be pseudoconformally equivalent to one of the three following domains:

1. a plane with the center deleted,
2. a ring \( 0 < r < |z| < 1 \),
3. a disc with the center deleted.

However, the Bergman function for the domain \( \{ z | z \neq 0 \} \) is identically zero, and the Bergman function for the ring possesses zeros by
Theorem 1. Since the class of Lu Qi-Keng domains is invariant under pseudoconformal transformations, (1) and (2) do not occur.

**Theorem 3.** For every $k \geq 3$, there exists a domain $D \subset \mathbb{C}$ of connectivity $k$ which is not a Lu Qi-Keng domain.

**Proof.** Consider a ring $R$ and let $z_0$, $i_0$ be such that $K_R(z_0, i_0) = 0$. We choose $k - 2$ distinct points $z_1, \ldots, z_{k-2}$ in $R$ different from $z_0$ and $i_0$. Consider a domain $R_m = R - \bigcup_{j=1}^{k-2} \{z \in R \text{ and } |z - z_j| \leq 1/m, m \text{ a positive integer}\}$. By the Ramadanov Theorem [3], the sequence $K_{R_m}(z, i_0)$ converges locally uniformly to $K_D(z, i_0)$ where $D = \bigcup_{m=1}^{\infty} R_m$. Since $K_D(z, i_0) \equiv K_R(z, i_0)$, we conclude that for sufficiently large $m$, the degree of connectivity of $R_m$ is $k$, and by Hurwitz’s theorem the function $K_{R_m}(z, i_0)$ has a zero in $R_m$.

**References**


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