FINDING A BOUNDARY FOR A 3-MANIFOLD

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In [1] Browder, Levine and Livesay considered the following problem: "Given an open manifold \( M \), when is it the interior of a compact manifold with boundary?" They were able to show that if dimension of \( M \geq 6 \) and if \( M \) was 1-connected at infinity, then a necessary and sufficient condition was that the homology of \( M \) be finitely generated. Edwards [4] and Wall [10] showed that if dimension of \( M \) was 3 and if \( M \) was 1-connected at infinity then \( M \) was homeomorphic to the interior of a compact manifold. Siebenmann [9] obtained necessary and sufficient conditions when dimension of \( M \geq 6 \). In this note, we prove the following.

**Theorem.** Let \( M \) be a connected, orientable 3-manifold with compact boundary and one end. The interior of \( M \) is homeomorphic to the interior of a compact 3-manifold if and only if there exists a positive integer \( N \) such that every compact subset of \( M \) is contained in the interior of a compact 3-manifold \( M' \) with connected boundary such that

1. \( \pi_1(M-M') \) is finitely generated;
2. genus (bdry \( M' \)) \( \leq N \);
3. every contractible 2-sphere in \( M-M' \) bounds a 3-cell.

**Remarks.** The referee has pointed out to the author that the boundary is unique by [12].

If the Poincaré Conjecture is true, hypothesis 3 is unnecessary. However if there is a counterexample to the Poincaré Conjecture it is possible to construct a counterexample to the theorem if 3 is not assumed. Whitehead's example [11] shows that 2 and 3 are not sufficient. It is unknown to the author whether 1 and 3 are sufficient. There is no loss of generality to assume that we are working in the piecewise linear category.

By van Kampen's Theorem [3, p. 71], it follows that \( \pi_1(M) \) is finitely generated. Hence it follows that \( M=\bigcup_{i=1}^{\infty} M_i \) such that for all \( i=1, 2, \ldots \)

1. \( M_i \) is a compact 3-manifold with connected boundary such that every contractible 2-sphere in \( M-M_i \) bounds a 3-cell;
2. \( M_i \subseteq \text{int} \ M_{i+1} \) (\( = \text{interior} \ M_{i+1} \));
3. \( \pi_1(M-M_i) \) is finitely generated;

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4. \( H_k(M, M_i) = 0 \) for \( k = 0, 1 \);

5. genus (bdry \( M_i \)) = \( n \) where \( n \) is an integer such that if \( M = \bigcup_{i=1}^{n} M_i \) where \( M_i \) satisfy 1–4, then genus \( M_i \geq n \) for all \( i \).

For all \( i \), let \( N_i = M_{i+1} - \text{int} M_i \). We have two cases to consider. If \( n = 0 \), it follows from \([6]\) that if \( \pi_1(N_i) \) is trivial, then \( N_i \) is homeomorphic to the product of the 2-sphere with the interval. Hence if all but a finite number of \( N_i \)'s have trivial fundamental group, we are through. If not, it follows from van Kampen's Theorem and \([5, \text{p. 192}]\) that \( \pi_1(M - M_i) \) is not finitely generated.

Let us now suppose that \( n \geq 1 \). We make the following claim.

6. The natural map \( \pi_i(\text{bdry } M_i) \to \pi_i(\bigcup_{i=1}^{n} N_k) \) is a monomorphism for all but a finite number of \( i \)'s.

Suppose that \( \pi_i(\text{bdry } M_i) \to \pi_i(\bigcup_{i=1}^{n} N_k) \) is not a monomorphism. By the Loop Theorem \([7]\), there exists a simple loop \( S \) on \( \text{bdry } M_i \) which is homotopically nontrivial on \( \text{bdry } M_i \) but \( S \) is contractible in \( \bigcup_{i=1}^{n} N_k \). It follows from Dehn's Lemma \([8]\) that \( S = \text{bdry } D \) where \( D \) is a 2-cell such that \( D \cap \text{bdry } M_i = S \). Let \( N \) be a regular neighborhood of \( D \) in \( \bigcup_{i=1}^{n} N_k \) which meets \( \text{bdry } M_i \) in a regular neighborhood of \( S \). Let \( M_i' = M_i \cup N \).

We have two possibilities. If \( S \) does not separate \( \text{bdry } M_i \), then \( M_i \) is a compact 3-manifold with connected boundary of genus \((n - 1)\). Suppose \( S \) separates \( \text{bdry } M_i \). Since \( H_k(M, M_i) = 0 \), \( k = 0, 1 \), then \( H_k(M, M_i') = 0 \), \( k = 0, 1 \). Hence \( H_0(\text{bdry } M_i') = 0 \), \( k = 0, 1 \). Therefore \( \pi_1(\text{bdry } M_i') = \pi_1(M_i') \) and \( M_i' \) has two components, one of which is bounded, say \( B \). Let \( M_i'' = M_i' \cup B \). Then \( M_i'' \) is a compact 3-manifold with connected boundary of genus \(< n \).

If \( \pi_i(\text{bdry } M_i) \to \pi_i(\bigcup_{i=1}^{n} N_k) \) is not a monomorphism for infinitely many \( i \)'s then \( M = \bigcup_{i=1}^{n} M_i \) where genus (bdry \( M_i \)) \(< n \) contradicting 5. Hence by reindexing we may assume

6'. For \( i = 1, 2, \cdots \), the natural map \( \pi_i(\text{bdry } M_i) \to \pi_i(\bigcup_{i=1}^{n} N_k) \) is a monomorphism.

7. There are only finitely many \( i \)'s such that there exists a surface \( S \) in \( N_i \) which separates \( \text{bdry } M_i \) from \( \text{bdry } M_{i+1} \) and which has genus \(< n \).

This claim follows from 5. Hence we have by reindexing

7'. If \( S \) is a surface in \( N_i, i = 1, 2, \cdots \), which separates \( \text{bdry } M_i \) from \( \text{bdry } M_{i+1} \), then genus \( S \geq n \).

8. For \( i = 2, 3, \cdots \), the natural map \( \pi_i(\text{bdry } M_i) \to \pi_i(\bigcup_{i=1}^{n} N_k) \) is a monomorphism.

The proof of 8 is by induction on \( i \). Suppose \( i = 2 \) and \( \pi_1(\text{bdry } M_2) \to \pi_1(N_1) \) is not one-one. Then as in the proof of 6, there exists a 2-cell \( D \) in \( N_1 \) such that \( D \cap \text{bdry } M_2 = \text{bdry } D \) is a nontrivial loop on \( M_2 \).
Again we have two cases to consider depending upon whether bdry $D$ separates bdry $M_2$.

If bdry $D$ does not separate bdry $M_2$, then we can easily find a surface in the interior of $N_1$ separating bdry $M_1$ from bdry $M_2$ of genus $<n$. This contradicts $7'$. Now consider the case when bdry $D$ separates bdry $M_2$.

Let $\overline{N}$ be a regular neighborhood of $D$ in $\Omega_1$ such that $\overline{N} \cap$ bdry $M_2$ is a regular neighborhood of bdry $D$ in bdry $M_2$. Let $M'_2 = M_2 - \text{int}(\overline{N} \cup N_2)$. Since $M_1$ can be chosen to contain the carriers of elements of $H_1(M)$, we have the following exact sequence

$$0 \to H_1(M, M'_2) \to H_0(M'_2) \to H_0(M) = 0.$$ 

But

$$H_1(M, M'_2) = H_1(M - \text{int} M'_2, \text{bdry} M'_2).$$

Again,

$$H_1(bdry M'_2) \to H_1(M - \text{int} M'_2) \to H_1(M - \text{int} M'_2, \text{bdry} M'_2) \to H_0(bdry M'_2) \to H_0(M - \text{int} M'_2) = 0.$$

bdry $D$ separates bdry $M_2$ implies that $H_0(bdry M'_2) = \mathbb{Z}$ (= integers). Since $H_1(M - \text{int} M'_2, \text{bdry} M'_2)$ maps onto $\mathbb{Z}$ and is free Abelian (for it is isomorphic to $H_0(M'_2)$), $H_1(M - \text{int} M'_2, \text{bdry} M'_2)$ has rank greater than zero. Hence $H_0(M'_2)$ has rank greater than zero and thereby the number of components of $M'_2$ is greater than one. The boundary components of $M'_2$ are $A_1$ and $A_2$ where $A_i \cap \text{bdry} M_2 \neq \emptyset$; in fact $A_1 \cap \text{bdry} M_2$ is separated from $A_2 \cap \text{bdry} M_2$ by bdry $D$.

Since $N_1$ is a connected compact manifold, it follows that the component $R_1$ of $M'_2$ which contains bdry $M_1$ also contains one of the $A_i$'s, say $A_1$. By using the collar of $A_1$ in $R$, one can find a homeomorphic copy of $A_1$, say $\overline{A}_1$, such that $\overline{A}_1$ separates bdry $M_1$ from $A_1$. Since $N_1$ is connected, there is only one other component of $M'_2$, say $R_2$, and bdry $R_2 = A_2$. Thus $\overline{A}_1$ separates bdry $M_1$ from bdry $M_2$ and genus $\overline{A}_1 < n$. This contradicts $7'$. Hence $\pi_1(bdry M_2) \to \pi_1(N_1)$ is a monomorphism.

The induction argument is essentially the same as the case $i = 2$, except that the disk $D$ may not lie in $N_i$; i.e., $D$ may intersect $R = \bigcup_{k=1}^{\infty} \text{bdry} M_k$. If so, put $D$ into general position with respect to $R$, keeping bdry $D$ fixed. Then $D \cap R$ is a finite collection of simple closed curves $\{S_i\}_{i=1}^n$. Pick an innermost $S_i$ on $D$; suppose $S_i$ bounds $D_j$ on $D$ and $S_i \subseteq \text{bdry} M_k$. Then $D_j$ lies either in $N_k$ or $N_{k-1}$. By either the induction hypothesis or $6'$, $S_i$ bounds a disk $D'_j$ on bdry $M_k$. Replace $D$ by $D' = (D - D_j) \cup D'_j$. By using the collar of bdry $M_k$ we may
"pop" $D_j$ off bdry $M_k$, eliminating the singularity $S_j$ without introducing any new singularities. After a finite number of these steps we finally get a disk $D'' \subseteq N_i$ such that $D'' \cap $bdry$ N_i =$bdry $D$. We continue then as in the case $i=2$ to get a contradiction.

9. If for $i=2, 3, \ldots$, the natural maps $\pi_1($bdry$ M_i) \rightarrow \pi_1(N_i)$ are epimorphisms, then the theorem is true.

This claim follows from the following fact. If $\pi_1($bdry$ M_i) \rightarrow \pi_1(N_i)$ is an epimorphism, then by [2], $N_i$ is homeomorphic to ($\text{bdry} M_i \times [0, 1]$. We are left to consider the possibility that for infinitely many $i$'s, $\pi_1($bdry$ M_i) \rightarrow \pi_1(N_i)$ is not onto. There is no loss of generality to assume that this occurs for all $i=2, 3, \ldots$. By van Kampen's theorem,

$$\pi_1\left( \bigcup_{i=1}^{k+1} N_i, p_{k+1} \right) = \pi_1\left( \bigcup_{i=1}^{k} N_i, p_{k+1} \right) \ast_{G_{k+1}} \pi_1(N_{k+1}, p_{k+1})$$

i.e., if $p_{k+1}$ is a point in bdry $M_{k+1}$, then the fundamental group of $\bigcup_{i=1}^{k+1} N_i$ is the free product of the fundamental groups of $\bigcup_{i=1}^{k} N_i$ and $N_{k+1}$ with amalgamated subgroup $G_{k+1} = \pi_1($bdry$ M_{k+1}, p_{k+1})$. It follows from 6' and 8 that the "natural map"

$$\phi: \pi_1\left( \bigcup_{i=1}^{k} N_i, p_{k+1} \right) \rightarrow \pi_1\left( \bigcup_{i=1}^{k+1} N_i, p_{k+1} \right)$$

is a monomorphism. Since $\pi_1($bdry$ M_{k+1}, p_{k+1}) \rightarrow \pi_1(N_{k+1}, p_{k+1})$ is not onto, $\phi$ is not onto. Hence we may assume that $\pi_1(\bigcup_{i=1}^{k} N_i, p_1)$ is identified with a proper subgroup of $\pi_1(\bigcup_{i=1}^{k+1} N_i, p_1)$ with "identifying map" $\phi_1$. Then $\pi_1(\bigcup_{i=1}^{k} N_i)$ is the direct limit of $\{ \pi_1(\bigcup_{i=1}^{k} N_i, p_1) \}$ and hence can be written as the infinite monotone union of proper subgroups which implies that $\pi_1(\bigcup_{i=1}^{k} N_i) = \pi_1(M-M)$ is infinitely generated. This contradiction establishes the theorem.

REFERENCES


**Florida State University and**
**University of Georgia**