WATTS COHOMOLOGY OF FIELD EXTENSIONS

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Let \( R \) be a commutative ring and \( A \) a commutative \( R \)-algebra. In [4] Watts defined a cohomology theory, \( H^*_R(A) \), which yields the Čech cohomology of the compact Hausdorff space \( X \) in the case when \( R = R \) and \( A = C(X) \), the ring of continuous real valued functions on \( X \). The definition of \( H^*_R(A) \) was in terms of a specific complex derived from the “additive Amitsur complex.” The question of the possible functorial significance of this cohomology theory was raised. As a step in this direction we compute here the Watts cohomology \( H^*_K(L) \), where \( K \) is a field and \( L \) is an arbitrary extension field of \( K \).

We recall the definition of \( H^*_K(L) \). The complex \( F^*_K(L) \) is the additive Amitsur complex [3] with a dimension shift of 1: \( F^*_K(L) \) is the \( n+1 \)-fold tensor product of \( L \) over \( K \), and the coboundary map 
\[
d^* : F^*_K(L) \to F^*_{K+1}(L)
\]

\[
d^*(x_0 \otimes \cdots \otimes x_n) = \sum_{i=0}^{n+1} (-1)^i x_0 \otimes \cdots \otimes x_{i-1} \otimes 1 \otimes x_i \otimes \cdots \otimes x_n.
\]

The homology of this complex is easily found.

**Proposition 1.** The complex \( F^*_K(L) \) has zero homology except in dimension zero, where \( H^0(F^*_K(L)) \cong K \).

**Proof.** It is known [3, Lemma 4.1] that the complex \( 0 \to K \to F^0_K(L) \to F^1_K(L) \cdots \) is acyclic.

Let \( \mu : F^*_K(L) \to L \) by \( \mu(x_0 \otimes \cdots \otimes x_n) = x_0 \cdots x_n \). The subcomplex \( N^*_K(L) \) is given by
\[
N^*_K(L) = \{ x \in F^*_K(L) \mid \exists y \in F^*_K(L) \text{ with } \mu_n(y) \neq 0 \text{ and } xy = 0 \}
\]
(the definition is simplified here by the fact that \( L \) is a field). Note that \( N^*_K(L) \subseteq \ker \mu \). The Watts cohomology \( H^*_K(L) \) is then defined to be the homology of the quotient complex \( C^*_K(L) = F^*_K(L)/N^*_K(L) \). Let \( \pi_n : F^*_K(L) \to C^*_K(L) \) denote the standard map.

Let \( L_s \) be the separable closure of \( K \) in \( L \). We shall prove the following

**Theorem.** The complexes \( C^*_K(L) \) and \( F^*_L(L_s) \) are canonically isomorphic.

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The following corollary is then immediate by Proposition 1.

**Corollary.** Watts cohomology for field extensions is given by $H^0_K(L) \cong L$, and $H^n_K(L) = 0$ for $n > 0$.

We now establish the theorem.

**Proposition 2.** Let $L$ be separable algebraic over $K$, and let $x \in F^n_K(L)$ with $\mu_n(x) = 0$. Then there is a $y \in F^n_K(L)$ with $\mu_n(y) \neq 0$ and $xy = 0$.

**Proof.** It suffices to consider the case where $L$ is a finite extension of $K$. But then $F^n_K(L)$ is a semisimple ring and every ideal is a direct factor. Hence $F^n_K(L) \cong L' \times \ker(\mu_n)$, where $L'$ is a field and $\mu_n$ maps $L'$ isomorphically onto $L$.

**Corollary.** If $L$ is separable algebraic over $K$, $N^n_K(L) = \ker(\mu_n)$, and there is a canonical isomorphism $\beta_n : C^n_K \to L$ such that $\beta_n \circ \tau_n = \mu_n$.

**Proposition 3.** Let $K$ be separably closed in $L$ (i.e. $L = K$) and let $A$ be a commutative $K$-algebra in which every zero divisor is nilpotent. Then every zero divisor in $A \otimes_K L$ is nilpotent.

**Proof.** We may work within a finitely generated subalgebra of $A$, and hence assume $A$ is Noetherian. Let $N$ be the ideal of nilpotents in $A$. Then $(0)$ is a primary ideal in $A$ and $N$ is its associated prime. It then follows [1, Chapter IV, §2.6, Theorem 2] (with $E = A$, $F = B = A \otimes_K L$) that the associated prime ideals of $(0)$ in $A \otimes_K L$ coincide with the associated prime ideals of the ideal $N \otimes_K L$ in $A \otimes_K L$. But by [2, Chapter IV, Theorem 24] every zero-divisor in $A/N \otimes_K L$ is nilpotent and hence $N \otimes_K L$ is a primary ideal.

**Corollary.** If $K$ is separably closed in $L$, then every zero divisor in $F^n_K(L)$ is nilpotent, for all $n$, and $F^n_K(L) = C^n_K(L)$.

Now let $\theta_n : F^n_K(L) \to F^n_{L^\alpha}(L)$ by $\theta_n(x_0 \otimes \cdots \otimes x_n) = x_0 \otimes \cdots \otimes x_n$. It is clear that $\theta_n$ is surjective and that $\theta = \{\theta_n\}$ is a map of complexes, $\theta : F^n_K(L) \to F^n_{L^\alpha}(L)$. Let $x \in N^n_K(L)$, with $y$ satisfying $xy = 0$, $\mu_n(y) \neq 0$. Then $\theta_n(x) \theta_n(y) = 0$ and $\mu_n(\theta_n(x)) = \mu_n(y) \neq 0$, so $\theta_n(x) \in N^n_{L^\alpha}(L) = (0)$. Hence $\theta$ induces a surjective map of complexes $\tau : C^n_K(L) \to C^n_{L^\alpha}(L) = F^n_{L^\alpha}(L)$. To complete the proof of the theorem we will construct an inverse to $\tau$.

Fix an integer $n$, and let $A$ denote the ring $F^n_K(L)$. Let $B$ be the subring $F^n_K(L)$. If $M$ is an $A$-module and $\rho : A \to M$ an $A$-linear map, then $\ker(\rho) \supset A(\ker(\rho) \mid \rho)$ and hence $\rho$ factors as $A = A \otimes_B B \to A \otimes_B \rho(B) \to M$. We apply this method to the two maps $\theta_n : A \to F^n_{L^\alpha}(L)$, and $\pi_n : A \to C^n_K(L)$. 


Proposition 4. The induced map $A \otimes_B \theta_n(B) = F^n_n(L) \otimes F^n_n(L) \rightarrow F^n_n(L)$ is an isomorphism.

Proof. First note that $\theta_n(B)$ is canonically isomorphic to $L_n$. We construct an inverse. Let $\omega: L \times \cdots \times L \rightarrow A \otimes_B \theta_n(B)$ by $\omega(x_0, \cdots, x_n) = x_0 \otimes \cdots \otimes x_n \otimes 1$. If $y \in L_n$, $x_0 \otimes \cdots \otimes y_{x_i} \otimes \cdots \otimes x_n \otimes 1 = x_0 \otimes \cdots \otimes x_i \otimes \cdots \otimes x_n \otimes y$. Hence $\omega$ is $L_n$-multilinear and induces $F^n_n(L) \rightarrow A \otimes_B \theta_n(B)$.

Consider the following diagram:

\[ \begin{array}{ccc}
A = A \otimes_B B & \rightarrow & A \otimes_B \theta_n(B) \rightarrow F^n_n(L) \\
\downarrow & & \downarrow \\
A \otimes_B C^n_n(L) & \rightarrow & C^n_n(L) \\
\downarrow & & \\
C^n_n(L) & & \\
\end{array} \]

Using Propositions 3 and 4, $\theta_n(B) \cong L_n \cong C^n_n(L)$ and hence we obtain, by Proposition 4,

\[ F^n_n(L) \rightarrow A \otimes_B \theta_n(B) \rightarrow A \otimes_B C^n_n(L) \rightarrow C^n_n(L), \]

and the composite map is easily seen to be inverse to $\tau_n$.

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References


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