THE ORDER OF THE ANTIPODE OF A HOPF ALGEBRA

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The purpose of this note is to show that the order of the antipode of a Hopf algebra is not necessarily 2, but can be any positive even integer or infinite.

A coalgebra over a field $K$ is a vector space $C$ over $K$ together with maps $\delta: C \to C \otimes C$ and $\varepsilon: C \to K$ satisfying

$$ (\delta \otimes 1)\delta = (1 \otimes \delta)\delta \quad \text{and} \quad (\varepsilon \otimes 1)\delta = (1 \otimes \varepsilon)\delta = 1. $$

The coalgebra is cocommutative if $t\delta = \delta$, where $t: C \otimes C \to C \otimes C$ is defined by $t(c \otimes d) = d \otimes c$. A Hopf algebra over $K$ is an associative algebra $H$ with identity, together with identity-preserving algebra morphisms $\delta: H \to H \otimes H$ and $\varepsilon: H \to K$ which give the underlying vector space a coalgebra structure. If $H$ is an associative algebra, we will denote by $\mu: H \otimes H \to H$ the map defined by $\mu(h \otimes k) = hk$, and by $\eta: K \to H$ the map defined by $\eta(a) = a1$. An antipode for the Hopf algebra $H$ is a map $\gamma: H \to H$ satisfying

$$ \mu(\gamma \otimes 1)\delta = \mu(1 \otimes \gamma)\delta = \eta\varepsilon. $$

If there exists an antipode $\gamma$ for the Hopf algebra $H$, then it is unique, and is a Hopf algebra antieomorphism. If $H$ is either commutative or cocommutative, then $\gamma^2 = 1$. (See §8 of [1] for proofs of these facts.) The order of the antipode $\gamma$ is the smallest positive integer $n$ such that $\gamma^n = 1$, if such an integer exists, and is infinite otherwise. Since $\gamma$ is an antieomorphism, if $\gamma$ has finite order it must have even order, unless $H$ is both commutative and cocommutative, in which case $\gamma$ may have order 1.

**Theorem.** If $n$ is a positive even integer or infinite, there exists a Hopf algebra over $R$ which has an antipode of order $n$.

**Free Hopf algebras over coalgebras.** If $X$ is a vector space over $K$, we denote by $T(X)$ the tensor algebra of $X$. Let $C$ with the maps $\delta, \varepsilon$ be a coalgebra. The map $C \to T(C) \otimes T(C)$ given by $c \to \delta(c)$ (where we are identifying $C \otimes C$ with a subspace of $T(C) \otimes T(C)$ by means of the usual identification of $C$ with a subspace of $T(C)$) induces an algebra morphism $\delta_T: T(C) \to T(C) \otimes T(C)$. Also the map $\varepsilon: C \to K$ induces an

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algebra morphism $e_T : T(C) \to K$. It is easily seen that $T(C)$ together
with the maps $\delta_T$ and $\epsilon_T$ is a Hopf algebra, called the free Hopf algebra
on $C$.

An antiautomorphism of the coalgebra $C$ is a bijective map $\xi : C \to C$
satisfying

$$
\delta_T = (\xi \otimes \xi) \delta \quad \text{and} \quad \epsilon_T = \epsilon.
$$

Given an antiautomorphism of $C$, consider the ideal $I \subseteq T(C)$ generated
by all elements of the form

$$
e_T(c)1 - \mu_T(\xi \otimes 1)\delta(c) \quad \text{and} \quad e_T(c)1 - \mu_T(1 \otimes \xi)\delta(c),
$$

where $c \in C \subseteq T(C)$. It is easily checked that $\delta(I) \subseteq T(C) \otimes I + I \otimes T(C)$
and that $\epsilon(I) = 0$. This implies that $H(C; \xi) = T(C)/I$ together with
the maps $\delta_H, \epsilon_H$ induced by $\delta_T, \epsilon_T$ is a Hopf algebra. If $\kappa$ is the unique
antiendomorphism of $T(C)$ satisfying $\kappa | C = \xi$, then $\kappa(I) \subseteq I$, so $\kappa$
induces a map $\gamma_H : H(C; \xi) \to H(C; \xi)$.

The facts that $\gamma_H$ is an algebra antiendomorphism and that

$$
\mu_H(\gamma_H \otimes 1)\delta_H(c + I) = \mu_H(1 \otimes \gamma_H)\delta_H(c + I) = \eta_H \epsilon_H(c + I)
$$

for all $c \in C$ imply that $\gamma_H$ is an antipode for the Hopf algebra $H(C; \xi)$. The Hopf algebra $H(C; \xi)$ is called the free Hopf algebra with antipode
on $C$ and $\xi$. Denote by $\pi : C \to H(C; \xi)$ the composition of the maps
$C \to T(C) \to H(C; \xi)$.

Warning. The map $\pi : C \to H(C; \xi)$ need not be injective.

The following Proposition is immediate:

**Proposition.** Let $C$ be a coalgebra, and let $\xi$ be an antiautomorphism
of $C$. Then there exist a Hopf algebra $H(C; \xi)$ with antipode $\gamma$, and a
morphism of coalgebras $\pi : C \to H(C; \xi)$ with $\pi \xi = \gamma \pi$, such that for every
Hopf algebra $H$ with antipode $\gamma'$ and every coalgebra morphism $f : C \to H$
satisfying $\gamma' f = f \xi$, there exists a unique Hopf algebra morphism
$g : H(C; \xi) \to H$ with $g \pi = f$.

**Construction of the example.** We prove the Theorem as follows: we
construct a coalgebra $C$ with antiautomorphism $\xi$ of order $n$ such that
$\pi : C \to H(C; \xi)$ is injective. Then $\gamma | \pi(C) = \pi \xi | \pi(C)$ is of order $n$,
so that $\gamma$ is of order at least $n$. On the other hand, it is clear that $\gamma$
is of order at most $n$.

To construct $C$ and $\xi$ we construct a finite dimensional algebra $A$
over $R$ with an antiautomorphism $\sigma$ of order $n$, and define $C = A^* = \text{hom}(A, R)$, $\delta = \mu'$, $\epsilon = \eta'$, and $\xi = \sigma'$. If $A$ has a basis $\{a_i\}$ and a
multiplication table $a_ia_j = \sum m_{ijk}a_k$, then $\delta$ is given explicitly by
$\delta(a^*_i) = \sum m_{ijk}a^*_j \otimes a^*_k$, where $\{a^*_i\}$ is the basis of $C$ dual to the basis
\{a_i\}. If 1 = \sum e_i a_i, then \(e(a_i^*) = e_i\). If \(\sigma(a_i) = \sum s_i a_j\), then \(\zeta(a_i^*) = \sum s_i a_j^*\). Thus, in this case \(H(C; \zeta)\) can be described as the associative algebra generated by \(\{a_i^*\}\), subject to the relations

\[ e_i = \sum m_{ik} s_j a_j a_k \quad \text{and} \quad e_i = \sum m_{ik} s_j a_j a_k. \]

To prove that \(\pi: C \to H(C; \zeta)\) is injective, it is sufficient to find a representation \(\rho: H(C, \zeta) \to \text{hom}(V, V)\) such that \(\{\rho(a_i^*)\}\) is linearly independent.

Let \(A\) be the algebra of all \(2 \times 2\) matrices over \(R\). If \(n\) is a positive even integer, let \(\theta = 2\pi/n\). If \(n\) is infinite, let \(\theta = \alpha\pi\), where \(\alpha\) is any irrational number. Let

\[ U = \begin{bmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{bmatrix}. \]

Define \(\sigma: A \to A\) by \(\sigma(T) = U^{-1} T^* U\), for all \(T \in A\). The matrices

\[ C' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad S' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \]

\[ X' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y' = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \]

are a basis for \(A\). With respect to this basis, \(\sigma(C') = C', \sigma(S') = -S', \sigma(X') = \cos \theta X' + \sin \theta Y', \text{ and } \sigma(Y') = -\sin \theta X' + \cos \theta Y'.\) It is clear that the antiautomorphism \(\sigma\) is of order \(n\).

Let \(C, S, X,\) and \(Y\) be the basis of \(C = A^*\) dual to the given basis of \(A\). We apply the discussion in the second paragraph of this section to show that \(H(C; \zeta)\) is the algebra generated by \(C, S, X,\) and \(Y,\) subject to the relations

1. \(1 = C^2 + S^2 + X\gamma(X) + Y\gamma(Y),\)
2. \(1 = C^2 + S^2 + \gamma(X) X + \gamma(Y) Y,\)
3. \(0 = -CS + SC + X\gamma(Y) - Y\gamma(X),\)
4. \(0 = CS - SC + \gamma(X) Y - \gamma(Y) X,\)
5. \(0 = C\gamma(X) + S\gamma(Y) + XC + YS,\)
6. \(0 = CX - SY + \gamma(X) C - \gamma(Y) S,\)
7. \(0 = C\gamma(Y) - S\gamma(X) - XS + YC,\)
8. \(0 = CY + SX + \gamma(X) S + \gamma(Y) C,\)

where \(\gamma(X) = \cos \theta X - \sin \theta Y\) and \(\gamma(Y) = \sin \theta X + \cos \theta Y.\)

Some straightforward calculations show that equations (1)-(4) are equivalent to

9. \(XY = YX,\)
10. \(1 = C^2 + S^2 + \cos \theta(X^2 + Y^2),\)
11. \(0 = CS - SC - \sin \theta(X^2 + Y^2),\)

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and that the equations (5)–(8) are equivalent to
(12) \(0 = CX + \cos \theta XC - \sin \theta XS\),
(13) \(0 = SX + \sin \theta XC + \cos \theta XS\),
(14) \(0 = CY + \cos \theta YC - \sin \theta YS\),
(15) \(0 = SY + \sin \theta YC + \cos \theta YS\).

Therefore \(H(C; \xi)\) is the algebra generated by \(C\), \(S\), \(X\), and \(Y\), subject to the slightly less formidable relations (9)–(15).

We now define the representation \(\rho\). Consider the representation of \(T(C)\) on a three dimensional vector space defined by

\[
\begin{align*}
C & \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, & S & \rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sin \theta & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
X & \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & Y & \rightarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\end{align*}
\]

Routine calculations show that this representation preserves the relations (9)–(15) and so induces a representation \(\rho\) of \(H(C; \xi)\) on this vector space. Clearly \(\rho(C)\), \(\rho(S)\), \(\rho(X)\) and \(\rho(Y)\) are linearly independent if \(n > 2\). Therefore the antipode of \(H(C; \xi)\) has order \(n\).

Q.E.D.

**Reference**