EXTENDING ULM’S THEOREM WITHOUT
GROUP THEORY

FRED RICHMAN AND ELBERT WALKER

1. Introduction. In 1933 Ulm [6] showed that a countable reduced
abelian \( p \)-group \( G \) was determined by a function \( F_\alpha \) from ordinals
to cardinals defined by \( F_\alpha = \dim p^\alpha G/p^{\alpha+1}G \), the Ulm
invariants of \( G \). In 1960 Kolettis [3] extended this result to direct
sums of such groups. Hill [1] has recently improved Kolettis’ work
by showing that if \( G_i \) and \( H_i \) are countable reduced abelian \( p \)-groups
such that \( G = \sum_{i \in I} G_i \) and \( H = \sum_{i \in I} H_i \) have the same Ulm invariants
then there exists a partition of \( I \) into countable subsets \( I_a \) such
that \( \sum_{i \in I_a} G_i \) and \( \sum_{i \in I_a} H_i \) have the same Ulm invariants. Ulm’s
theorem then implies that \( G \) and \( H \) are isomorphic.

Hill’s work uses the fact that if \( G \) is a direct sum of countable
reduced \( p \)-groups then the socle of \( G \) is decomposable, i.e. \( G[p] = \sum_{\lambda < \lambda(\Omega)} K_\lambda \) where every nonzero element of \( K_\lambda \) has height
precisely \( \lambda \) in \( G \). This is a nontrivial group theoretic property, whereas
one would suspect that Hill’s theorem holds on purely set theoretic
grounds.

There is more than esthetics involved in eliminating the require-
ment of the decomposability of \( G[p] \). Let \( C \) be the class of totally
projective \( p \)-groups [4] of length less than \( \Omega \omega \) where \( \Omega \) and \( \omega \) are,
respectively, the first uncountable and the first infinite ordinals.
Parker and Walker [5] have shown that two groups in \( C \) are iso-
morphic if and only if they have the same Ulm invariants. This gen-
eralizes Kolettis’ theorem since \( C \) contains the class of direct sums
of countable reduced \( p \)-groups [4]. In proving this generalization
the following situation arises: \( G = \sum_{i \in I} G_i \) and \( H = \sum_{i \in I} H_i \) have
the same Ulm invariants, \( G_i \) and \( H_i \) are totally projective with \( |G_i|, |H_i| \leq \aleph_1 \), and it is necessary to partition \( I \) into subsets \( I_a \) such that
\( |I_a| \leq \aleph_1 \) and \( \sum_{i \in I_a} G_i \) and \( \sum_{i \in I_a} H_i \) have the same Ulm invariants.
Hill’s procedure cannot be followed because the socles of \( G \) and \( H \)
need not be decomposable. In fact [2], if \( G[p] \) is decomposable
then \( pG = \{0\} \). Again, such a partition of \( I \), if it exists, should exist
on purely set theoretic grounds. This is indeed the case and our pur-
pose here is to establish the relevant set theoretic facts which im-
mediately yield both this and the theorem of Hill.

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2. **The theorem.** We separate part of the construction into a lemma.

**Lemma.** Let \( m \) be an infinite cardinal number. Let \( f \) be a function from the set \( X \) to the cardinal numbers. Suppose

\[
\sum_{x \in X} f(x) = \sum_{x \in X} g(x) \quad \text{for all } x \in X
\]

where \( \sum_{x \in X} f(x) \leq m \leq \sum_{x \in X} g(x) \) for all \( \lambda \in I \), and \( m < |I| \). If \( S \subseteq I \), \( |S| < |I| \), then there exists a set \( S \supseteq S \) such that

\[
\sum_{\lambda \in S} f(x) = \sum_{\lambda \in S} g(x), \quad \sum_{x \in S} f(x) = \sum_{x \in S} g(x)
\]

and

**Case 1.** \( |I| \) is not the sum of \( \aleph_0 \) smaller cardinals: \( |S| < |I| \) and if for some \( x \in X \), \( f(x) < |I| \), \( \sum_{x \in X} f(x) \neq 0 \) and \( f(x) \) or \( g(x) \neq 0 \), then \( \lambda \in S \).

**Case 2.** \( |I| \) is a sum of \( \aleph_0 \) smaller cardinals: \( |S| \leq m |S| \).

**Proof.** Construct \( S = S_1 \cup T_1 \cup U_1 \cup S_2 \cup T_2 \cup \cdots \) so that

(a) \( \sum_{x \in S} f(x) \geq \sum_{x \in T} g(x) \geq \sum_{x \in S} f(x) \).

(b) Case 1: \( |S_1| < |I| \), Case 2: \( |S_1| \leq m |S| \).

(c) \( S_{i+1} = U_j \cup \lambda \) \( \forall x \in X \), \( \sum_{x \in U_j} f(x) \neq 0 \) and \( f(x) \) or \( g(x) \neq 0 \) and, Case 1: \( f(x) < |I| \), Case 2: \( f(x) \leq m |S| \).

Let \( S = S_j \). Since \( S = S_j \cup T_j = U U_j \) it follows from (a) that

\[
\sum_{x \in S} f(x) \geq \sum_{x \in S} g(x).
\]

In Case 1, since \( |S_1| < |I| \), then \( |S| < |I| \); in Case 2, since \( |S_1| \leq m |S| \) we have \( |S| \leq m |S| \). Suppose

\[
\sum_{x \in S \setminus S} f(x) \neq \sum_{x \in S \setminus S} g(x)
\]

for some \( x \in X \). Since \( f(x) = \sum_{x \in S} f(x) + \sum_{x \in S \setminus S} f(x) = \sum_{x \in S} g(x) + \sum_{x \in S \setminus S} g(x) \) and \( \sum_{x \in S \setminus S} f(x) = \sum_{x \in S \setminus S} g(x) \) is, in Case 1, less than \( |I| \) and, in Case 2, less than or equal to \( m |S| \), we must have, in Case 1, that \( f(x) < |I| \) and, in Case 2, that \( f(x) \leq m |S| \). Hence by (c) every \( \lambda \) such that \( f(x) \) or \( g(x) \neq 0 \) must be in \( S \). But this implies that \( \sum_{x \in S \setminus S} f(x) = \sum_{x \in S \setminus S} g(x) = 0 \). The second condition in Case 1 follows immediately from (c).

**Theorem.** Let \( m \) be an infinite cardinal number. Let \( f \) be a function from the set \( X \) to the cardinal numbers such that

\[
f(x) = \sum_{x \in I} f(x) = \sum_{x \in I} g(x) \quad \text{for all } x \in X,
\]

where \( \sum_{x \in X} f(x) \leq m \leq \sum_{x \in X} g(x) \) for all \( \lambda \in I \). Then there exists a partition of \( I \) into subsets \( S_\alpha \) of cardinality \( \leq m \) such that

\[
\sum_{x \in S_\alpha} f(x) = \sum_{x \in S_\alpha} g(x).
\]

**Proof.** We show that if \( |I| > m \) then such a partition can be found.
where $|S_a| < |I|$. By induction we will have proved the theorem.

Initially well order $I = \{i_\alpha\}$, $\alpha < |I|$.

**Case 1.** Define sets $S_\alpha \subseteq I$ for ordinals $\alpha < |I|$ as follows: Let

$$S_\alpha = \{i_\beta \in \bigcup_{i_\gamma} S_i \mid \beta = \alpha \text{ or } \exists \gamma > \beta, i_\gamma \in \bigcup_{i_\delta < \alpha} S_{i_\delta}\}$$

where the "bar" is taken inside $I \setminus \bigcup_{i_\alpha < \alpha} S_i$. Observe that $|S_\alpha| < |I|$ and $|\bigcup_{i_\alpha < \alpha} S_i| = |I|$ only if $\bigcup_{i_\alpha < \alpha} S_i = I$. Also,

$$\sum_{\lambda \in I \setminus \bigcup_{i_\alpha < I} S_i} f_\lambda(x) = \sum_{\lambda \in I \setminus \bigcup_{i_\alpha < I} S_i} g_\lambda(x),$$

for otherwise $f(x) < |I|$ and hence $f_\lambda(x) = g_\lambda(x) = 0$ for all $\lambda \in I \setminus \bigcup_{i_\alpha < I} S_i$ or for all $\lambda \in \bigcup_{i_\alpha < I} S_i$. Thus the construction makes sense and gives the desired partition.

In Case 2, let $\alpha_1 < \alpha_2 < \alpha_3 < \cdots$ be a sequence of ordinals with limit $|I|$. Define

$$S_j = \{i_\beta \in \bigcup_{i_\gamma} S_i \mid \beta < \alpha_j\}$$

where the "bar" is taken inside $I \setminus \bigcup_{i_\gamma < I} S_i$. The $S_j$ give the desired partition.

**Corollary (Hill).** If $G = \sum_{\lambda \in I} A_\lambda$ and $H = \sum_{\lambda \in I} B_\lambda$ are reduced $p$-groups with the same Ulm invariants, $|A_\lambda| \equiv \aleph_0 \equiv |B_\lambda|$, then there is a partition of the set $I$ into countable sets $S_\alpha$ such that

$$\sum_{\lambda \in S_\alpha} A_\lambda \cong \sum_{\lambda \in S_\alpha} B_\lambda.$$

**Proof.** Apply the theorem to the Ulm invariants of $G$ and use Ulm's theorem.

**Bibliography**


*New Mexico State University*