The purpose of this note is to establish that for topological lattices of suitably small breadth, connectedness implies modularity without an exploitation of compactness. $L$ will denote a topological lattice, that is a Hausdorff topological space with continuous binary operations $\lor$ and $\land$ for which $(L, \lor, \land)$ is a lattice. For a more explicit presentation and some related machinery, see the paper of E. Dyer and A. Shields [5]. Of specific need, $L$ is a modular lattice iff for every $a, b, c \in L$, $b \lor a$ implies $a \land (b \lor c) = b \lor (a \land c)$. And if $n$ is a positive integer, the breadth of $L$ is less than $n$ means that for every $x_1, x_2, \cdots, x_n \in L$, there exists $j$ such that $x_j \leq \lor_{i \neq j} x_i$. Note that the breadth formulation given here is dual to the equivalent phrasing used in [5]. Also for $a \leq b$ in $L$, define $[a, b] = \{x \in L \mid a \leq x \leq b\}$.

**Theorem 1.** If $L$ is a connected topological lattice and the breadth of $L$ is less than three, then $L$ is modular.

**Proof.** It suffices to show that $L$ does not contain a nonmodular five lattice [4]. Thus let $a, b, c \in L$ such that $b \lor a$, $a \land c = b \land c$, and $a \lor c = b \lor c$. Under the hypotheses of the theorem it will be established that $b = a$, and therefore $L$ cannot contain a nonmodular five lattice.

For $x$ such that $a \leq x \leq a \lor c$ and $a \land c \neq x \land c$, let

$$A = \{y \in [c, c \lor a] \mid a \land y \leq b \lor (x \land c)\}.$$

Clearly $A \cup \{c \lor a\} \subset [c, c \lor a]$. Now if $c \leq y < c \lor a$, the breadth condition implies (1) $a \land y \leq b \lor (x \land c)$, (2) $b \leq (a \land y) \lor (x \land c)$, or (3) $x \land c \leq b \lor (a \land y)$. But (2) implies $b \leq (a \land y) \lor c \leq y$ and hence $a \land c = b \lor c \leq y$, contrary to $y < a \land c$. Also (3) implies $x \land c = a \land c$, contrary to assumption. Thus if $c \leq y < c \lor a$, then $y \in A$ and $A \cup \{c \lor a\} = [c, c \lor a]$. But $A$ and $\{c \lor y\}$ are closed nonempty sets whose union is a connected set since $L$ is connected. Thus $c \lor a \in A$ and $a = a \land (c \lor a) \leq b \lor (x \land c)$. Therefore $[a, a \lor c] = B \cup C$ where

$$B = \{x \in [a, a \lor c] \mid a \land c = x \land c\}$$

and

$$C = \{x \in [a, a \lor c] \mid a \leq b \lor (x \land c)\}.$$
Immediately $B$ and $C$ are nonempty closed sets and since $[a, a \lor c]$ is connected, $B \cap C \neq \emptyset$. Therefore there exist $d$ such that $a \land c = d \land c$ and $a \leq b \lor (d \land c)$, and $a \leq b \lor (d \land c) = b \lor (a \land c) = b \lor (b \land c) = b$. Thus $a = b$.

**Theorem 2.** If $L$ is a locally compact connected topological lattice of codimension less than three, $L$ is modular.

**Proof.** Lawson [7] has shown that a locally compact connected topological lattice of codimension $n$ has breadth less than or equal to $n$. This generalizes the result of L. W. Anderson and L. E. Ward, Jr. [1] Thus the theorem follows from Theorem 1.

**Remark.** In an earlier paper [6] the author gave an example of a compact connected topological lattice of dimension three which is nonmodular. This example is seen to have breadth three also, thus in some respects the result above is the best possible.

**References**


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