Let $L$ be the lattice of all ideals of a ring with unity and let $X$ be a subset of the set of all real ideals of $A$ [1] which is equipped with the Stone topology [4, p. 272]. Anderson [1, Lemma 4, 5(2)] gave necessary and sufficient conditions, in terms of certain elements of $L$, for two subsets of $X$ to be completely separated in $X$. In [6, §8], we modified the idea of Anderson in case $L$ is the lattice of all $S_\alpha$-ideals of a $C$-lattice [2, §§2 and 3] for $\alpha$ a real number and $X$ a certain subset of $L$ equipped with the Stone topology. The purpose of this note is to give necessary and sufficient conditions for two subsets of $X$ to be completely separated in $X$ in case $L$ is an arbitrary complete lattice and $X$ is a subset of $L$ which is completely regular when equipped with the Stone topology. The idea is essentially that of Anderson in [1]. We use this result to complete the “internal” characterization of the $\Phi$-algebra of all real-valued continuous functions on an arbitrary completely regular space for which Henriksen and Johnson [6] have results in special cases.

Suppose now that $L$ is a complete lattice and $X$ is a fixed subset of $L$. We define a function $\phi$ from $L$ into $L$ by

\[ \phi(x) = \bigwedge \{ y \in X \mid x \leq y \} \quad (x \in L) \]

and we let $K = \{ x \in L \mid x = \phi(x) \}$. It is clear that $\phi(x)$ is merely the meet of the intersection of $X$ with the principal dual ideal of $L$ generated by $x$. It is also a routine matter to verify that $\phi$ is a closure operation [3, p. 49] on $L$ and hence by [8, Theorem 4.1], $K$ is a complete lattice relative to the partial order on $L$. Moreover, meets in $K$ coincide with meets in $L$ and joins in $K$ are obtained by operating on the joins in $L$ by $\phi$. In addition, $X \subseteq K$, $\Lambda X = \Lambda K$, and if $1 = \bigvee L$, then $1 \in K$.

We now suppose that $X$ is also equipped with the Stone topology. Using the fact that the closure of a subset $Y$ of $X$ is $\{ y \in X \mid \Lambda Y \leq y \}$, one can show that the mapping $F \rightarrow \Lambda F$ is a dual isomorphism of the lattice of all closed subsets of $X$ onto $K$.

Next, we define a binary relation $*$ on $K$ as follows: $x * y$ in case
there exists $z \in K$ such that $z \wedge x = \wedge X$ and $z \vee y = 1$. The motivation for this definition is the following: If $F_1$ and $F_2$ are closed subsets of $X$, then $F_2 \cap F_1$ if and only if $\wedge F_1 \ast \wedge F_2$. Note that this implies that $\ast$ is a transitive relation on $K$.

If $C \subseteq K$, we say that $C$ is a dense chain with respect to $\ast$ in Case (i) for each $x, y \in C$, either $x \ast y$ or $y \ast x$, and (ii) if $x, y \in C$ with $x \ast y$, then there exists $z \in C$ such that $x \ast z \ast y$. We now define a binary relation $\perp$ on $K$ by $x \perp y$ in case there exists a countable subset $C$ of $K$ such that $C$ is a dense chain with respect to $\ast$, $x \leq \wedge C$, and $\forall C \leq y$.

**Theorem 1.** Let $X$ be a subset of a complete lattice $L$ so that $X$ is a completely regular space when equipped with the Stone topology. Subsets $A_1$ and $A_2$ of $X$ are completely separated in $X$ if and only if there exist closed subsets $F_1$ and $F_2$ of $X$ such that $A_1 \subseteq X \setminus F_1$, $A_2 \subseteq F_2$, and $\wedge F_1 \perp \wedge F_2$.

**Proof.** Let $Q$ denote the rational numbers. It is not difficult to show that subsets $A_1$ and $A_2$ of a completely regular space $X$ are completely separated in $X$ if and only if there exist closed sets $P_i$ and $P_2$ of $X$ and a family $(U_t)_{t \in Q}$ of open sets of $X$ such that $A_1 \subseteq X \setminus F_1 \subseteq \bigcap_{t \in Q} U_t$, $\bigcup_{t \in Q} U_t \subseteq X \setminus F_2 \subseteq X \setminus A_2$, and $r < s$ in $Q$ implies that $\text{cl } U_r \subseteq U_s$.

Suppose first that $A_1$ and $A_2$ are completely separated in $X$, and let $F_1$, $F_2$, and $(U_t)_{t \in Q}$ be as described above. For each $t \in Q$, let $F_t = X \setminus U_t$. One can show that $\{\wedge F_t \mid t \in Q\}$ is a countable dense chain with respect to $\ast$ such that $\wedge F_1 \leq \wedge \{\wedge F_t \mid t \in Q\}$ and $\forall \{\wedge F_t \mid t \in Q\} \leq \wedge F_2$.

Conversely, if there exist closed sets $F_1$ and $F_2$ of $X$ such that $A_1 \subseteq X \setminus F_1$, $A_2 \subseteq F_2$, and $\wedge F_1 \perp \wedge F_2$, then there exists a countable subset $C$ of $K$ such that $C$ is a dense chain with respect to $\ast$, $\wedge F_1 \leq \wedge C$, and $\forall C \leq \wedge F_2$. The conditions on $C$ imply that there exists a function $f : Q \to C$ such that $r < s$ in $Q$ implies that $f(r) \ast f(s)$. For each $t \in Q$, let $U_t$ be the open subset of $X$ defined by $\wedge (X \setminus U_t) = f(t)$. It is a routine matter to verify that $(U_t)_{t \in Q}$ is a family of open sets satisfying $X \setminus F_1 \subseteq \bigcap_{t \in Q} U_t \subseteq X \setminus F_2$, and $r < s$ in $Q$ implies that $\text{cl } U_r \subseteq U_s$. Hence the theorem.

Suppose now that $A$ is a $\Phi$-algebra. We refer the reader to [5] for definitions, symbols, and notation not defined here. Let $L$ be the complete lattice of all $l$-ideals of $A$, let $\mathfrak{a}(A)$ be the set of all real maximal $l$-ideals of $A$ equipped with the Stone topology, and let $K(A)$ be the subset of $L$ as defined earlier using the operator $\phi$ and the subset $\mathfrak{a}(A)$ of $L$. It follows that $I \in K(A)$ if and only if
I = \bigcap \{ M \in \mathfrak{a}(A) \mid I \subseteq M \}. If \bigcap \mathfrak{a}(A) = \{ 0 \}, the binary relation * can be restated slightly as follows: \( I_1 * I_2 \) in case there exists \( J \in K \) with \( I_1 \cap J = \{ 0 \} \) and \( I_1 \cup J \subseteq M \) for any \( M \in \mathfrak{a}(A) \).

We can now obtain an "internal" characterization as a \( \Phi \)-algebra of the set \( C(X) \) of all real-valued continuous functions on an arbitrary completely regular space \( X \).

**Theorem 2 (Compare [5, 5.2]).** A \( \Phi \)-algebra \( A \) is isomorphic to \( C(X) \) for some completely regular space \( X \) if and only if

(i) \( A \) is an algebra of real-valued functions,

(ii) \( A \) is uniformly closed,

(iii) \( A \) is closed under inversion, and

(iv) for each pair \( I \perp J \) in \( K(A) \), there exists \( f \in A \) such that \( f - 1 \in J \) and \( f \wedge |h| = 0 \) for all \( h \in I \).

**Proof.** The only condition which needs comment concerning the necessity of the conditions is (iv). It follows easily from 4.6 in [5] and the fact that \( I \perp J \) in \( K(A) \) if and only if the corresponding sets in \( \mathfrak{a}(A) \) are completely separated.

In view of the proof of 5.2 in [5], the sufficiency of the four conditions will be immediate if it is shown that disjoint zero-sets in \( \mathfrak{a}(A) \) have disjoint closures in \( \mathfrak{m}(A) \), where \( \mathfrak{m}(A) \) is the space of all maximal \( l \)-ideals of \( A \). Let \( Z_1 \) and \( Z_2 \) be disjoint zero-sets in \( \mathfrak{a}(A) \). Then there exist closed subsets \( F_1 \) and \( F_2 \) in \( \mathfrak{a}(A) \) such that \( Z_1 \subseteq \mathfrak{a}(A) \setminus F_1 \), \( Z_2 \subseteq F_2 \), and \( \cap F_1 \perp \cap F_2 \). By (iv), there exists \( f \in A \) such that \( f - 1 \in \cap F_2 \) and \( f \wedge |h| = 0 \) for all \( h \in F_1 \). Thus, if \( M \in Z_1 \), then \( M \subseteq F_1 \) and so \( \cap F_1 \subseteq M \). Hence there exists \( h \in \cap F_1 \) such that \( h \in M \). Since \( M \) is an \( l \)-ideal, \( |h| \in \mathfrak{m}(A) \). By [7, 1, 3.15 (iii)], \( f \wedge |h| = 0 \) implies that \( f |h| = 0 \) and so \( f |h| \subseteq M \). By the primeness of \( M \), \( f \in M \) and thus \( \tilde{f}(M) = 0 \).

It is easy to see that if \( M \in Z_2 \), then \( \tilde{f}(M) = 1 \). Moreover, we may assume that \( 0 \leq \tilde{f}(M) \leq 1 \) for all \( M \in \mathfrak{a}(A) \) since \( (f / 0) \perp 1 \in A \). Thus \( \tilde{f}(M) \in R \) for all \( M \in \mathfrak{m}(A) \) and so \( \{ M \in \mathfrak{m}(A) \mid \tilde{f}(M) = 0 \} \) and \( \{ M \in \mathfrak{m}(A) \mid \tilde{f}(M) = 1 \} \) are disjoint closed sets in \( \mathfrak{m}(A) \) containing \( Z_1 \) and \( Z_2 \), respectively.

**References**


University of Florida and Mathematisch Centrum, Amsterdam