A THEOREM ON ONE-TO-ONE MAPPINGS ONTO THE PLANE

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Let X be a generalized continuum, i.e. let X be a connected, locally compact, separable metric space, and let F be a one-to-one continuous map of X onto the plane. A point $x \in X$ will be called a cut-point (respectively, a local separating point) of X provided x separates X (respectively, x separates some open set in X). See [7, pp. 41, 61]. By an m-manifold, $m \geqq 0$, we shall mean a separable metric space such that every point has a neighborhood homeomorphic to an open subset of a closed m-cell. All terms not defined here may be found in [7]. There are several known conditions on X that imply that F is a homeomorphism. In [4] L. C. Glaser proved that if X is a 2-manifold that satisfies several strong conditions, then F is a homeomorphism. In [2] E. Duda proved that if X is a locally connected generalized continuum that has the property that the complement of any compact set has exactly one component with a noncompact closure, then F is a homeomorphism. In [1] the author used Duda's result to show that if X is the product of two noncompact, locally connected, generalized continua, then F is a homeomorphism. In [5] D. H. Pettey was able to remove the conditions imposed by Glaser and prove that if X is any 2-manifold, F is a homeomorphism. There are several examples of one-to-one continuous maps of 3-manifolds onto E^3 which are not homeomorphisms. See [6] or Lemma 2 of [3]. It is an open question as to whether every one-to-one continuous map of a locally connected generalized continuum onto the plane is a homeomorphism. The purpose of this note is to prove the following:

THEOREM. *If X is a locally connected generalized continuum without any local separating points and F is a one-to-one continuous map of X onto the plane, then X is a 2-manifold and F is a homeomorphism.*

PROOF. Let $x \in X$. Since x does not separate any open set in X, x does not separate X and by Proposition (4.15) of [7, p. 50] there exists a connected open set V of X containing x such that $\overline{V}$ is compact and $X-V$ is connected. By [7, Proposition (15.43), p. 22] there exists a connected open set W containing x such that $\overline{W} \subset V$ and $\overline{W}$ is locally connected. Furthermore, since no point of W separates W, $\overline{W}$ has no cut-points. Since $F(\overline{W})$ is a subset of $F(V)$ and since

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$F(X-V)$ is an unbounded connected set, $F(X-V)$ lies entirely in the unbounded complementary domain of $F(\overline{W})$. Thus if H denotes the union of $F(\overline{W})$ and all of the bounded complementary domains of $F(\overline{W})$, H is a closed, compact and connected subset of $F(V)$ that contains the closed 2-cell bounded by any simple closed curve in H. Every complementary domain of $F(\overline{W})$ has no cut-points and $F(\overline{W})$ has no cut-points, and since the boundary of every complementary domain of $F(\overline{W})$ is a nondegenerate subset of $F(\overline{W})$, H has no cut-points. Since $F(\overline{W})$ is locally connected, it follows from [7, Theorem (4.4), p. 113] that H is locally connected. Thus H is a compact, connected, locally connected subset of the plane that has no cut-points and contains the closed 2-cell bounded by any simple closed curve in H; and hence, by the characterization of the closed 2-cell in [2], H is a closed 2-cell. Since $F|\overline{V}$ is a homeomorphism and since $H\subset V$, $B=F^{-1}(\overline{H})$ is a closed 2-cell in X and x is interior to B. Hence X is a 2-manifold, and by Theorem 2 of [5], F is a homeomorphism.

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