CONTINUOUS BOUNDARY VALUES OF ANALYTIC FUNCTIONS OF SEVERAL COMPLEX VARIABLES

BARNET M. WEINSTOCK[1]

1. Let D be a relatively compact domain in C^n, $n > 1$, with smooth boundary Γ. Let $A(\Gamma)$ be the algebra of continuous, complex-valued functions on Γ which have analytic extensions to D. In this note we derive the following characterization of $A(\Gamma)$:

THEOREM. *A continuous function f on Γ belongs to $A(\Gamma)$ if and only if*

$$\int_\Gamma f\omega = 0 \tag{1}$$

for each $(n, n-1)$ form ω which is C^∞ in $D \cup \Gamma$ and satisfies $\bar\partial\omega = 0$ in D.

Our method is elementary, and was suggested by the proofs of the Hartogs' Extension Theorem given by Bochner [1] and Martinelli [6]. We remark that the differential forms ω appearing in the theorem are a natural generalization to several variables of the "analytic differentials" which play the corresponding role in the well-known one-variable theorem. By appealing to the regularity theorem of Kohn [4] for the $\bar\partial$-Neumann problem we obtain as a corollary that if the Levi form on Γ has at least one positive eigenvalue at each point, then $f \in A(\Gamma)$ if and only if f is a weak solution of the tangential Cauchy-Riemann equations. (With this hypothesis on Γ, it follows that if D is connected, then so is Γ.)

This theorem was obtained by Fichera [2] under the additional hypotheses that Γ is connected and f is the boundary value of a function with finite Dirichlet integral. His proof is based on an approximation theorem for harmonic functions in the Dirichlet norm. Kohn and Rossi [5] have shown that the theorem holds for C^∞ functions when D is a finite domain on a complex manifold and the Levi form satisfies the above condition on Γ. They also obtain extension theorems for (p, q) forms.

2. We assume a basic familiarity with the algebra of complex differential forms. If S is an open set in C^n we denote by $E^{p,q}(S)$ the space of C^∞ forms of bidegree (p, q) on S. The usual exterior differential operator, d, splits as the direct sum $\partial + \bar\partial$, where $\partial: E^{p,q} \to E^{p+1,q}$ differentiates with respect to $z_1, \cdots, z_n$, and $\bar\partial: E^{p,q} \to E^{p,q+1}$ differ-

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entiates with respect to $\bar{z}_1, \cdots, \bar{z}_n$. We recall that the Hodge star operator maps $E^{p,q}$ isomorphically onto $E^{n-q,n-p}$. For 1-forms we have the formulas

$$*dz_k = -2(i/2)^n dz_k \wedge \omega_k, \qquad *d\bar{z}_k = 2(i/2)^n d\bar{z}_k \wedge \omega_k$$

where $\omega_k = \prod_{i\neq k} dz_i \wedge d\bar{z}_i$. If $g \in C^\infty(S)$ it follows that

$$*dg - 2*\partial g = 2(i/2)^n dg \wedge \sum_{k=1}^n \omega_k.$$

In particular if S is a neighborhood of the compact $(2n-1)$-manifold Γ, and $g=0$ on Γ, then the $(2n-1)$-forms $*dg$ and $2*\partial g$ define the same measure on Γ.

Every continuous function f on Γ has a harmonic extension to D given by

$$f(\zeta) = -(n-2)!\pi^{-n} \int_\Gamma f(z) * dG(z, \zeta)$$

where $G(z, \zeta)$ is the Green's function for D. Since for fixed $\zeta \in D$ we have $G(z, \zeta) = 0$, and also that $H(z, \zeta) = G(z, \zeta) - |z-\zeta|^{2-2n}$ is harmonic throughout D, we can write

$$\begin{aligned} \frac{1}{2} f(\zeta) &= -(n-2)!\pi^{-n} \int_\Gamma f(z)(*\partial\, |z-\zeta|^{2-2n} + *\partial H(z, \zeta)) \\ &= (-1)^n (n-1)!(2\pi i)^{-n} \int_\Gamma f(z)\, |z-\zeta|^{-2n} \sum_{i=1}^n (\bar{z}_i - \bar{\zeta}_i) dz_i \wedge \omega_i \qquad (2) \\ &\quad - (n-2)!\pi^{-n} \int_\Gamma f(z) * \partial H(z, \zeta) \end{aligned}$$

where, for each ζ, $\bar{\partial} * \partial H(z, \zeta) = 0$.

Let $\omega_{ij} = \prod_{k\neq i,j} dz_k \wedge d\bar{z}_k$ and

$$\Omega_j(\zeta) = \sum_{i=1}^n |z-\zeta|^{-2n} (\bar{z}_i - \bar{\zeta}_i) dz_j \wedge dz_i \wedge \omega_{ij}.$$

Since $\sum_{i=1}^n \partial(\bar{z}_i - \bar{\zeta}_i)|z-\zeta|^{-2n}/\partial\bar{z}_i = 0$, a straightforward computation shows that for $z \in D - \{\zeta\}$,

$$\bar{\partial}\Omega_j(\zeta) = -\sum_{i=1}^n \frac{\partial}{\partial \bar{z}_j} (\bar{z}_i - \bar{\zeta}_i)\, |z-\zeta|^{-2n} dz_i \wedge \omega_i.$$

For each $\zeta \in D$ we denote by $\Omega_j'(\zeta)$ a form in $E^{n,n-2}(D)$ which agrees with $\Omega_j(\zeta)$ in a neighborhood of Γ.

PROOF OF THEOREM. Necessity follows from Stokes' Theorem, since if $f\in A(\Gamma)$ and $\bar\partial\omega=0$ in D, then

$$\int_\Gamma f\omega = \int_D d(f\omega) = \int_D \bar\partial(f\omega) = \int_D \bar\partial f\wedge\omega + f\bar\partial\omega = 0.$$

(Here we denote again by f the analytic extension of f to D.) To prove the converse, extend f to a harmonic function in D. Since f satisfies (1), the second integral in (2) vanishes, hence the harmonic extension is given by the Bochner-Martinelli formula, which is the first integral in (2). But then, for each j, we have

$$\frac{1}{2}\frac{\partial f}{\partial\bar\zeta_j}(\zeta) = (-1)^n(n-1)!(2\pi i)^{-n}\int_\Gamma f(z)\bar\partial\Omega_j'(\zeta).$$

Since $\bar\partial\Omega_j'(\zeta)$ is a smooth, $\bar\partial$-closed $(n, n-1)$-form on $D\cup\Gamma$ for each ζ, the integral vanishes by (1), hence f is analytic. Q.E.D.

3. **Definition.** Let g be a C^∞ function defined in a neighborhood N of Γ such that:

(i) $\Gamma=\{g=0\}$.

(ii) $dg\neq 0$ on Γ.

(iii) $D\cap N=\{g<0\}$.

We say that Γ is 1-*strongly-pseudoconvex* if at each point of Γ the hermitian form $\sum\partial^2g/\partial z_i\partial\bar z_j a_i\bar a_j$ on the subspace defined by $\sum a_j\partial g/\partial z_j=0$ has at least one positive eigenvalue.

If $f\in C^\infty(\Gamma)$, let f' denote an arbitrary C^∞ extension of f to a neighborhood of Γ. If the tangential component (in terms of a "boundary coordinate system") of $\bar\partial f'$ vanishes on Γ we say that f is a *solution of the tangential Cauchy-Riemann* (C-R) *equations*. (See [5].) Equivalently, f is a solution of the tangential C-R equations if $X(f')=0$ on Γ for all complex vector fields X of the form $\sum\phi_j\partial/\partial\bar z_j$ which are tangent to Γ.

Let f be a solution of the tangential C-R equations and μ be a smooth $(n, n-2)$- form in a neighborhood of Γ. Then

$$\int_\Gamma f\bar\partial\mu = \int_\Gamma \bar\partial f'\wedge\mu + f\bar\partial\mu = \int_\Gamma \bar\partial(f'\mu) = 0.$$

If f is a continuous function on Γ we say that f is a *weak solution* of the tangential Cauchy-Riemann equations if the first integral above vanishes for all such μ.

COROLLARY. *If Γ is 1-strongly-pseudoconvex, then $f\in A(\Gamma)$ if and only if f is a weak solution of the tangential C-R equations.*

PROOF. With this assumption on Γ it follows that $H^{n-1}(D, \Omega^n)=0$ where Ω^n is the sheaf of germs of holomorphic n-forms on D. (See [3].) But by the Kohn regularity theorem [4]

$$H^{n-1}(D, \Omega^n) \approx \frac{\{\omega \in E^{n,n-1}(D \cup \Gamma): \bar\partial\omega = 0\}}{\{\partial\mu: \mu \in E^{n,n-2}(D \cup \Gamma)\}}.$$

Thus for each ω in the hypothesis of the theorem there is an $(n, n-2)$-form μ which is C^∞ in $D \cup \Gamma$ such that $\omega = \bar\partial\mu$.

We remark that if the domain D in $\mathbf{C}^n$ is connected and Γ is 1-strongly-pseudoconvex, then Γ is connected (cf. [5, Corollary 7.3]).

4. It is reasonable to conjecture that the theorem remains true for relatively compact domains on other complex manifolds, e.g., on Stein manifolds. For domains D on Kähler manifolds (see [7] for definition) and C^∞ boundary values f there is the following simple proof:

We can extend f to a harmonic function in D, i.e., a function satisfying $d^*df=0$. Because of the Kähler metric this implies that f also satisfies $\bar\partial^*\partial\bar f=0$. (See [7].) Hence $\int_\Gamma f^*\partial\bar f=0$. But

$$\int_\Gamma f^*\partial\bar f = \int_D \bar\partial f \wedge {}^*\overline{\bar\partial f}$$

and

$$\|\mu\| = \left(\int_D \mu \wedge {}^*\bar\mu\right)^{1/2} \qquad \mu \in E^{p,q}(D \cup \Gamma)$$

is a norm on $E^{p,q}(D \cup \Gamma)$. Hence $\bar\partial f=0$ on D, i.e., f is analytic.

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BROWN UNIVERSITY