NORM PRESERVING EXTENSIONS OF LINEAR TRANSFORMATIONS ON HILBERT SPACES

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Introduction. Let H be a Hilbert space and let D be a closed proper subspace of H. Let J_0 be a linear contraction on D to H. The problem of characterizing the contractions on all of H which extend J_0 is directly related to the extension problems for unbounded transformations posed and treated by M. G. Krein [2] and R. S. Phillips [3]. In §1 of this paper we establish the following solution of this problem: Let P be the orthogonal projection of H on D and $J_1 = J_0P$. The contractions on all of H which extend J_0 are precisely the transformations of the form $J_1 + (I - J_1J_1^*)^{1/2}B$, where B is any contraction which annihilates D. §2 is a short outline of the relationship of this result to the extension problem for dissipative transformations treated by Phillips in [3] and [4]. §3 contains some remarks concerning the case where J_0 is symmetric and one asks for the selfadjoint contractions which extend J_0, which is essentially the situation treated by Krein [2]. In particular, we obtain the Krein extension of a symmetric contraction to a selfadjoint contraction by, we feel, a less elegant but more illuminating argument than Krein's original one.

1. **The representation theorem.** All transformations considered here are linear. The domain, range and null space of a transformation L will be denoted by $D(L)$, $R(L)$, and $N(L)$ respectively. Our goal is to prove the following theorem:

THEOREM I. *Let H be a Hilbert space. Let D be a closed proper subspace of H and let J_0 be a contraction on D to H. Let P be the orthogonal projection of H on D and set*

$$J_1 = J_0P. \tag{1.1}$$

Then the contractions J on H to H which extend J_0 are precisely the transformations of the form

$$J = J_1 + (I - J_1J_1^*)^{1/2}B \tag{1.2}$$

where B is any contraction on H which annihilates D.

REMARK. The correspondence (1.2) between contractions J which

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extend J_0 and contractions B which annihilate D becomes biunique if we further require that $R(B)\subset[R(T)]$, where we have set

$$T = (I - J_1J_1^*)^{1/2}, \tag{1.3}$$

and $[R(T)]$ denotes the closure of $R(T)$.

PROOF OF THEOREM I. For convenience we set

$$E(J_0) = \{J: J \text{ is a contraction on all of } H \text{ and } J \text{ extends } J_0\}. \tag{1.4}$$

Let $J\in E(J_0)$. Clearly then $JP=J_1=J_0P$ and we have

$$J = JP + J(I - P) = J_1 + J(I - P) \qquad \text{for } J \text{ in } E(J_0). \tag{1.5}$$

The decomposition (1.5) shows that every element of $E(J_0)$ has the form J_1+K where K is a contraction annihilating D. Conversely, any transformation of this form extends J_0. Hence $E(J_0)$ consists of those sums J_1+K, where K is a contraction annihilating D, which are contractions. The theorem is therefore an immediate consequence of the following lemma.

LEMMA I. *Let the assumptions of Theorem* I *be satisfied. Then, if K is a contraction on H which annihilates D, J_1+K is a contraction if and only if K has the form $K=TB$, where T is given by* (1.3), *for some contraction B which annihilates D.*

PROOF OF LEMMA I. An attempt to prove Lemma I directly would lead to some difficulties, however, passing to the adjoint allows a trivial proof. We first show the sufficiency. Let B be a contraction annihilating D. J_1+TB is a contraction if $J_1^*+B^*T$ is a contraction (note that T is selfadjoint). Moreover, $N(J_1)\supset D^\perp$ (the orthogonal complement of D in H) and $N(B)\supset D$, so $R(J_1^*)$ and $R(B^*)$ are orthogonal. Hence, since B^* is a contraction and since $\|Tw\|^2=\|w\|^2-\|J_1^*w\|^2$ for w in H by (1.3), where $\|\ \|$ is the norm in H, we have

$$\begin{aligned}\|(J_1^* + B^*T)w\|^2 &= \|J_1^*w\|^2 + \|B^*Tw\|^2 \leqq \|J_1^*w\|^2 + \|Tw\|^2\\ &= \|J_1^*w\|^2 + \|w\|^2 - \|J_1^*w\|^2 = \|w\|^2\end{aligned}$$

for w in H, and $J_1^*+B^*T$ is a contraction.

Conversely, if K annihilates D and J_1+K is a contraction, then $R(J_1^*)$ and $R(K^*)$ are orthogonal and $J_1^*+K^*$ is a contraction, so we have

$$\|(J_1^* + K^*)w\|^2 = \|J_1^*w\|^2 + \|K^*w\|^2 \leqq \|w\|^2 \qquad \text{for } w \text{ in } H. \tag{1.6}$$

It follows from (1.6) that

$$\|K^*w\|^2 \leqq \|Tw\|^2 \qquad \text{for } w \text{ in } H. \tag{1.7}$$

According to (1.7) the map L on $R(T)$ to $R(K^*)$ defined by $LTw = K^*w$ is a contraction. We extend L to $[R(T)]$ by continuity and then to all of H by setting it equal to zero on $R(T)^\perp$. Then we have $LT = K^*$ and $K = TL^*$. Since $R(L) \subset R(K^*) \subset D^\perp$, $N(L^*) \supset D$ and, setting $B = L^*$, K has the form claimed by the lemma. This completes the proof of Lemma I and therefore Theorem I is also established.

REMARK. The proof of Theorem I may easily be adapted to provide a proof of the more general result stated below.

THEOREM I′. *Let H_1 and H_2 be Hilbert spaces. Let D be a closed proper subspace of H_1 and let J_0 be a bounded transformation on D to H_2. Let P be the orthogonal projection of H_1 on D and set $J_1 = J_0P$. Then the bounded transformations on H_1 to H_2 of norm not exceeding $k\|J_0\|$, $k \geqq 1$, which extend J_0 are precisely the transformations of the form $J_1 + (k^2\|J_0\|^2 - J_1J_1^*)^{1/2}B$, where B is a contraction on H_1 to H_2 which annihilates D.*

2. **The extension problem for dissipative transformations.** Let A be a transformation on $D(A) \subset H$ into H. A is called *dissipative* if

$$(u, Au) + (Au, u) \leqq 0 \qquad \text{for } u \text{ in } D(A). \tag{2.1}$$

If A is dissipative and has no proper dissipative extensions, it is called *maximal dissipative*. We consider the problem of finding all the maximal dissipative transformations which extend a given dissipative A. If A is dissipative then, in view of (2.1),

$$\begin{aligned} \|(I - A)u\|^2 &= \|u\|^2 + \|Au\|^2 - 2\,\mathrm{Re}(u, Au) \\ &\geqq \|u\|^2 + \|Au\|^2 \geqq \|u\|^2 + \|Au\|^2 + 2\,\mathrm{Re}(u, Au) \\ &= \|(I + A)u\|^2 \quad \text{for } u \text{ in } D(A). \end{aligned} \tag{2.2}$$

It follows from the first inequality of (2.2) that $I - A$ is one-to-one and that $R(I - A)$ is a closed subspace of H if A is a closed transformation, and from the inequality of the extreme members in (2.2) it follows that the transformation J_0 defined by

$$D(J_0) = R(I - A), \qquad J_0(I - A)u = (I + A)u \quad \text{for } u \text{ in } D(A) \tag{2.3}$$

is a contraction. J_0 is called the Cayley transform of A. If $D(A)$ is dense in H, then there is a one-to-one correspondence between the maximal dissipative extensions of A and the elements of $E(J_0)$, that is the contractions which extend J_0 to H. Given J in $E(J_0)$ the corresponding maximal dissipative extension of A is the dissipative transformation A_J whose Cayley transform is J: It is defined by

$$D(A_J) = R(I + J), \qquad A_J(I + J)u = (J - I)u \quad \text{for } u \text{ in } H. \tag{2.4}$$

For the straightforward proof of this see [3]. Theorem I, therefore, allows us to characterize the maximal dissipative extensions of a given dissipative transformation A. We proceed with this below.

Let A be a closed densely defined dissipative transformation and let J_0 be the Cayley transform of A. Let P be the orthogonal projection of H on $D(J_0)=R(I-A)$ and let $J_1=J_0P$. J_1 is determined by A as follows: $R(I-A)$ is closed and $R(I-A)^{\perp}=N(I-A^*)$. Every element w of H has a unique decomposition into a sum $w=(I-A)u+v$ with u in $D(A)$ and v in $N(I-A^*)$ and we have

$$(2.5)\quad J_1((I-A)u+v) = (I+A)u \text{ for } u \text{ in } D(A) \text{ and } v \text{ in } N(I-A^*).$$

J_1^* is related in a somewhat similar way to A^*. Since $N(I-A)=\{0\}$ and $R(I-A)$ is closed, $R(I-A^*)=H$. We claim

$$(2.6)\qquad J_1^*(I-A^*)u = P(I+A^*)u \quad \text{for } u \text{ in } D(A^*).$$

In view of (2.5), the validity of (2.6) is equivalent to

$$((I+A)u, (I-A^*)w) = ((I-A)u+v, P(I+A^*)w)$$
$$\text{for } u \text{ in } D(A), \quad w \text{ in } D(A^*) \quad \text{and} \quad v \text{ in } N(I-A^*),$$

which is true in view of

$$((I-A)u+v, P(I+A^*)w)$$
$$= (P((I-A)u+v), (I+A^*)w) = ((I-A)u, (I+A^*)w)$$

for u in $D(A)$, w in $D(A^*)$ and v in $N(I-A^*)$.

We have now, as an immediate corollary of Theorem I,

COROLLARY I. *Let A be a closed densely defined dissipative transformation. Then there is a one-to-one correspondence between contractions B on $N(I-A^*)$ to $[R(T)]$ where T is given by* (1.3), *J_1 by* (2.5) *and J_1^* by* (2.6), *and the maximal dissipative extensions of A. Given a contraction B on $N(I-A^*)$ to $[R(T)]$ the corresponding extension A_B of A is defined by*

$$D(A_B) = D(A) + (I+TB)N(I-A^*),$$
$$A_B(u+(I+TB)v) = Au + (TB-I)v$$
$$\textit{for } u \textit{ in } D(A) \quad \textit{and} \quad v \textit{ in } N(I-A^*).$$

REMARK 1. The main shortcoming of this result with regard to applications lies in the difficulty of computing the action of the transformation T given the transformation A. With further restrictions on the transformation A more convenient descriptions of its maximal dissipative extensions can be given. See M. G. Crandall and R. S. Phillips [1].

REMARK 2. A more general problem, including the extension problem for dissipative transformations, is treated by Phillips [3], [4] and [5]. In Theorem 5.3 of [4] Phillips gives a solution of this problem, the formulation of which involves completing a certain Hilbert space in a new metric. We remark that Theorem I′ may be applied directly to this more general problem, giving a new solution which does not involve any "completions."

3. **The symmetric case.** We adopt here the notations of §1. We now assume that J_0 is symmetric, i.e.,

$$(J_0u, v) = (u, J_0v) \qquad \text{for } u \text{ and } v \text{ in } D, \tag{3.1}$$

and ask for those J in $E(J_0)$ which are selfadjoint. See Krein [2] for the relationship of this question to the problem of characterizing the semibounded selfadjoint extensions of semibounded symmetric transformations. Let $J_1 = J_0P$ as before. Clearly J_0 is symmetric if and only if PJ_1 is selfadjoint. Let $J = J_1 + TB$ be an element of $E(J_0)$; B is a contraction annihilating D and T is given by (1.3). It is not easy to characterize those B for which J is selfadjoint, but we can easily characterize those B for which J^* extends J_0, and if J and J^* both extend J_0 then $(J+J^*)/2$ is a selfadjoint extension of J_0. We let $P_1 = (I-P)$; P_1 is the orthogonal projection of H on $D^\perp$. Then

$$\begin{aligned}(J_1 + TB)^* &= ((P + P_1)(J_1 + TB)(P + P_1))^* \\ &= PJ_1 + PJ_1^*P_1 + B^*TP + B^*TP_1,\end{aligned}$$

because PJ_1 is selfadjoint and $J_1P_1 = BP = 0$ by assumption, and it follows that $(J_1+TB)^*$ extends J_0 if and only if

$$PJ_1 + B^*TP = J_1$$

or

$$B^*TP = P_1J_1. \tag{3.2}$$

Equation (3.2) determines the action of B^* on $[TD]$. As noted below, (3.2) is consistent with the requirement that B^* be a contraction and Theorem I can be used to characterize those contractions B for which (3.2) is satisfied, which we leave to the reader. Here we restrict ourselves to noting that if L is defined by

$$\begin{aligned} L^*Tu &= P_1J_1u \qquad &&\text{for } u \text{ in } D \\ L^*v &= 0 &&\text{for } v \text{ in } (TD)^\perp \end{aligned} \tag{3.3}$$

then L^* (and hence L) is a contraction for $J_1^*P = (PJ_1)^* = PJ_1$ implies

$$\begin{aligned}\|Tu\|^2 &= \|u\|^2 - \|J_1^*u\|^2 = \|u\|^2 - \|J_1^*Pu\|^2 = \|u\|^2 - \|PJ_1u\|^2 \\ &= \|u\|^2 - \|J_1u\|^2 + \|P_1J_1u\|^2 \\ &\geqq \|P_1J_1u\|^2 \qquad \text{for } u \text{ in } D,\end{aligned}$$

and $R(L^*)\subset D^\perp$ implies $N(L)\supset D$. By the above arguments it follows that $(J_1+J_1^*+TL+L^*T)/2$ is a selfadjoint contraction which extends J_0; it is the extension found by Krein [2].

REMARK. If W is a bounded selfadjoint transformation on H and

$$J_0W = \varepsilon WJ_0 \qquad \text{for } \varepsilon = 1 \quad \text{or} \quad \varepsilon = -1, \tag{3.4}$$

it was noticed by Phillips [5, Lemma 3.1] that the Krein extension of J_0 will also satisfy (3.4) in place of J_0. Our construction makes this obvious, for if (3.4) holds then W commutes with P and P_1, so J_1 satisfies (3.4) and therefore J_1^* satisfies (3.4) and therefore $(I-J_1J_1^*)^{1/2}$ commutes with W and finally the transformations L^* and L defined by (3.3) satisfy (3.4). It follows that the Krein extension of J_0 will also satisfy (3.4).

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