NORM PRESERVING EXTENSIONS OF LINEAR
TRANSFORMATIONS ON HILBERT SPACES

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Introduction. Let $H$ be a Hilbert space and let $D$ be a closed proper
subspace of $H$. Let $J_0$ be a linear contraction on $D$ to $H$. The problem
of characterizing the contractions on all of $H$ which extend $J_0$ is
directly related to the extension problems for unbounded transforma-
tions posed and treated by M. G. Krein [2] and R. S. Phillips [3]. In
§1 of this paper we establish the following solution of this problem:
Let $P$ be the orthogonal projection of $H$ on $D$ and $J_1 = J_0 P$. The
contractions on all of $H$ which extend $J_0$ are precisely the transforma-
tions of the form $J_1 + (I - J_1 J_1^*)^{1/2} B$, where $B$ is any contraction which
annihilates $D$. §2 is a short outline of the relationship of this result to
the extension problem for dissipative transformations treated by
Phillips in [3] and [4]. §3 contains some remarks concerning the case
where $J_0$ is symmetric and one asks for the selfadjoint contractions
which extend $J_0$, which is essentially the situation treated by Krein
[2]. In particular, we obtain the Krein extension of a symmetric
contraction to a selfadjoint contraction by, we feel, a less elegant but
more illuminating argument than Krein's original one.

1. The representation theorem. All transformations considered
here are linear. The domain, range and null space of a transformation
$L$ will be denoted by $D(L)$, $R(L)$, and $N(L)$ respectively. Our goal is
to prove the following theorem:

**Theorem I.** Let $H$ be a Hilbert space. Let $D$ be a closed proper
subspace of $H$ and let $J_0$ be a contraction on $D$ to $H$. Let $P$ be the orthogonal
projection of $H$ on $D$ and set

\[(1.1) \quad J_1 = J_0 P.\]

Then the contractions $J$ on $H$ to $H$ which extend $J_0$ are precisely the
transformations of the form

\[(1.2) \quad J = J_1 + (I - J_1 J_1^*)^{1/2} B\]

where $B$ is any contraction on $H$ which annihilates $D$.

**Remark.** The correspondence (1.2) between contractions $J$ which
extend $J_0$ and contractions $B$ which annihilate $D$ becomes biunique
if we further require that $R(B) \subset [R(T)]$, where we have set

$$T = (I - J_0J_0^*)^{1/2},$$

and $[R(T)]$ denotes the closure of $R(T)$.

**Proof of Theorem I.** For convenience we set

$$(1.4) \quad E(J_0) = \{J: J \text{ is a contraction on all of } H \text{ and } J \text{ extends } J_0\}.$$

Let $J \in E(J_0)$. Clearly then $JP = J_1 = J_0P$ and we have

$$(1.5) \quad J = JP + J(I - P) = J_1 + J(I - P) \quad \text{for } J \in E(J_0).$$

The decomposition (1.5) shows that every element of $E(J_0)$ has the form $J_1 + K$ where $K$ is a contraction annihilating $D$. Conversely, any transformation of this form extends $J_0$. Hence $E(J_0)$ consists of those sums $J_1 + K$, where $K$ is a contraction annihilating $D$, which are contractions. The theorem is therefore an immediate consequence of the following lemma.

**Lemma I.** Let the assumptions of Theorem I be satisfied. Then, if $K$ is a contraction on $H$ which annihilates $D$, $J_1 + K$ is a contraction if and only if $K$ has the form $K = TB$, where $T$ is given by (1.3), for some contraction $B$ which annihilates $D$.

**Proof of Lemma I.** An attempt to prove Lemma I directly would lead to some difficulties, however, passing to the adjoint allows a trivial proof. We first show the sufficiency. Let $B$ be a contraction annihilating $D$. $J_1 + TB$ is a contraction if $J_1^* + B^*T$ is a contraction (note that $T$ is selfadjoint). Moreover, $N(J_1) \supset D^\perp$ (the orthogonal complement of $D$ in $H$) and $N(B) \supset D$, so $R(J_1^*)$ and $R(B^*)$ are orthogonal. Hence, since $B^*$ is a contraction and since $\|Tw\|^2 = \|w\|^2 - \|J_1^*w\|^2$ for $w$ in $H$ by (1.3), where $\|$ is the norm in $H$, we have

$$\|(J_1^* + B^*T)w\|^2 = \|J_1^*w\|^2 + \|B^*Tw\|^2 \leq \|J_1^*w\|^2 + \|Tw\|^2 = \|J_1^*w\|^2 + \|w\|^2 - \|J_1^*w\|^2 = \|w\|^2,$$

for $w$ in $H$, and $J_1^* + B^*T$ is a contraction.

Conversely, if $K$ annihilates $D$ and $J_1 + K$ is a contraction, then $R(J_1^*)$ and $R(K^*)$ are orthogonal and $J_1^* + K^*$ is a contraction, so we have

$$(1.6) \quad \|(J_1^* + K^*)w\|^2 = \|J_1^*w\|^2 + \|K^*w\|^2 \leq \|w\|^2 \quad \text{for } w \text{ in } H.$$

It follows from (1.6) that

$$(1.7) \quad \|K^*w\|^2 \leq \|Tw\|^2 \quad \text{for } w \text{ in } H.$$
According to (1.7) the map $L$ on $R(T)$ to $R(K^*)$ defined by $LTw = K^*w$ is a contraction. We extend $L$ to $[R(T)]$ by continuity and then to all of $H$ by setting it equal to zero on $R(T)^\perp$. Then we have $LT = K^*$ and $K = TL^*$. Since $R(L) \subset R(K^*) \subset D^\perp$, $N(L^*) \supset D$ and, setting $B = L^*$, $K$ has the form claimed by the lemma. This completes the proof of Lemma I and therefore Theorem I is also established.

Remark. The proof of Theorem I may easily be adapted to provide a proof of the more general result stated below.

Theorem I. Let $H_1$ and $H_2$ be Hilbert spaces. Let $D$ be a closed proper subspace of $H_1$ and let $J_0$ be a bounded transformation on $D$ to $H_2$. Let $P$ be the orthogonal projection of $H_1$ on $D$ and set $J_1 = J_0P$. Then the bounded transformations on $H_1$ to $H_2$ of norm not exceeding $k||J_0||$, $k \geq 1$, which extend $J_0$ are precisely the transformations of the form $J_1 + (k^2||J_0||^2 - J_1J_1^*)^{1/2}B$, where $B$ is a contraction on $H_1$ to $H_2$ which annihilates $D$.

2. The extension problem for dissipative transformations. Let $A$ be a transformation on $D(A) \subset H$ into $H$. $A$ is called dissipative if

$$\langle u, Au \rangle + \langle Au, u \rangle \leq 0 \quad \text{for } u \in D(A).$$

If $A$ is dissipative and has no proper dissipative extensions, it is called maximal dissipative. We consider the problem of finding all the maximal dissipative transformations which extend a given dissipative $A$. If $A$ is dissipative then, in view of (2.1),

$$\| (I - A)u \| = \| u \| + \| Au \| - 2 \Re \langle u, Au \rangle$$

(2.2)

$$\geq \| u \|^2 + \| Au \| \geq \| u \|^2 + \| Au \|^2 + 2 \Re \langle u, Au \rangle = \| (I + A)u \|^2 \quad \text{for } u \in D(A).$$

It follows from the first inequality of (2.2) that $I - A$ is one-to-one and that $R(I - A)$ is a closed subspace of $H$ if $A$ is a closed transformation, and from the inequality of the extreme members in (2.2) it follows that the transformation $J_0$ defined by

$$D(J_0) = R(I - A), \quad J_0(I - A)u = (I + A)u \quad \text{for } u \in D(A)$$

(2.3)

is a contraction. $J_0$ is called the Cayley transform of $A$. If $D(A)$ is dense in $H$, then there is a one-to-one correspondence between the maximal dissipative extensions of $A$ and the elements of $E(J_0)$, that is the contractions which extend $J_0$ to $H$. Given $J$ in $E(J_0)$ the corresponding maximal dissipative extension of $A$ is the dissipative transformation $A_J$ whose Cayley transform is $J$: It is defined by

$$D(A_J) = R(I + J), \quad A_J(I + J)u = (J - I)u \quad \text{for } u \in H.$$

(2.4)
For the straightforward proof of this see [3]. Theorem I, therefore, allows us to characterize the maximal dissipative extensions of a given dissipative transformation $A$. We proceed with this below.

Let $A$ be a closed densely defined dissipative transformation and let $J_0$ be the Cayley transform of $A$. Let $P$ be the orthogonal projection of $H$ on $D(J_0) = R(I-A)$ and let $J_1 = J_0P$. $J_1$ is determined by $A$ as follows: $R(I-A)$ is closed and $R(I-A)^\perp = N(I-A^*)$. Every element $w$ of $H$ has a unique decomposition into a sum $w = (I-A)u + v$ with $u$ in $D(A)$ and $v$ in $N(I-A^*)$ and we have

$$J_1((I-A)u + v) = (I + A)u \quad \text{for} \quad u \in D(A) \quad \text{and} \quad v \in N(I-A^*).$$

$J_1^*$ is related in a somewhat similar way to $A^*$. Since $N(I-A) = \{0\}$ and $R(I-A)$ is closed, $R(I-A^*) = H$. We claim

$$J_1^*(I-A^*)u = P(I+A^*)u \quad \text{for} \quad u \in D(A^*).$$

In view of (2.5), the validity of (2.6) is equivalent to

$$((I+A)u, (I-A^*)w) = ((I-A)u + v, P(I+A^*)w)$$

for $u$ in $D(A)$, $w$ in $D(A^*)$ and $v$ in $N(I-A^*)$.

We have now, as an immediate corollary of Theorem I,

**Corollary I.** Let $A$ be a closed densely defined dissipative transformation. Then there is a one-to-one correspondence between contractions $B$ on $N(I-A^*)$ to $[R(T)]$ where $T$ is given by (1.3), $J_1$ by (2.5) and $J_1^*$ by (2.6), and the maximal dissipative extensions of $A$. Given a contraction $B$ on $N(I-A^*)$ to $[R(T)]$ the corresponding extension $A_B$ of $A$ is defined by

$$D(A_B) = D(A) + (I+TB)N(I-A^*),$$

$$A_B(u + (I+TB)v) = Au + (TB - I)v$$

for $u$ in $D(A)$ and $v$ in $N(I-A^*)$.

**Remark 1.** The main shortcoming of this result with regard to applications lies in the difficulty of computing the action of the transformation $T$ given the transformation $A$. With further restrictions on the transformation $A$ more convenient descriptions of its maximal dissipative extensions can be given. See M. G. Crandall and R. S. Phillips [1].
Remark 2. A more general problem, including the extension problem for dissipative transformations, is treated by Phillips [3], [4] and [5]. In Theorem 5.3 of [4] Phillips gives a solution of this problem, the formulation of which involves completing a certain Hilbert space in a new metric. We remark that Theorem $I'$ may be applied directly to this more general problem, giving a new solution which does not involve any “completions.”

3. The symmetric case. We adopt here the notations of §1. We now assume that $J_0$ is symmetric, i.e.,

$$ (J_0 u, v) = (u, J_0 v) \quad \text{for } u \text{ and } v \text{ in } D, $$

and ask for those $J$ in $E(J_0)$ which are selfadjoint. See Krein [2] for the relationship of this question to the problem of characterizing the semibounded selfadjoint extensions of semibounded symmetric transformations. Let $J_1 = J_0 P$ as before. Clearly $J_0$ is symmetric if and only if $P J_1$ is selfadjoint. Let $J = J_1 + TB$ be an element of $E(J_0)$; $B$ is a contraction annihilating $D$ and $T$ is given by (1.3). It is not easy to characterize those $B$ for which $J$ is selfadjoint, but we can easily characterize those $B$ for which $J^*$ extends $J_0$, and if $J$ and $J^*$ both extend $J_0$ then $(J + J^*)/2$ is a selfadjoint extension of $J_0$. We let $P_1 = (I - P); P_1$ is the orthogonal projection of $H$ on $D^\perp$. Then

$$ (J_1 + TB)^* = ((P + P_1)(J_1 + TB)(P + P_1))^* $$

$$ = PJ_1 + PJ_1^* P_1 + B^* TP + B^* TP_1, $$

because $PJ_1$ is selfadjoint and $J_1 P_1 = BP = 0$ by assumption, and it follows that $(J_1 + TB)^*$ extends $J_0$ if and only if

$$ PJ_1 + B^* TP = J_1 $$

or

$$ (3.2) \quad B^* TP = P_1 J_1. $$

Equation (3.2) determines the action of $B^*$ on $[TD]$. As noted below, (3.2) is consistent with the requirement that $B^*$ be a contraction and Theorem I can be used to characterize those contractions $B$ for which (3.2) is satisfied, which we leave to the reader. Here we restrict ourselves to noting that if $L$ is defined by

$$ L^* Tu = P_1 J_1 u \quad \text{for } u \text{ in } D $$

$$ L^* v = 0 \quad \text{for } v \text{ in } (TD)^\perp $$

then $L^*$ (and hence $L$) is a contraction for $J_1^* P = (PJ_1)^* = PJ_1$ implies...
\[ ||Tu||^2 = ||u||^2 - ||J^*_1u||^2 = ||u||^2 - ||J^*_1Pu||^2 = ||u||^2 - ||PJ^*_1u||^2 \]
\[ = ||u||^2 - ||J^*_1u||^2 + ||PJJ^*_1u||^2 \]
\[ \geq ||PJJ^*_1u||^2 \quad \text{for } u \text{ in } D, \]

and \( R(L^*) \subset D^+ \) implies \( N(L) \supset D \). By the above arguments it follows that \( (J_1 + (J^*_1 + TL + L^*T)^2)/2 \) is a selfadjoint contraction which extends \( J_0 \); it is the extension found by Krein [2].

**Remark.** If \( W \) is a bounded selfadjoint transformation on \( H \) and

\[ J_0W = \varepsilon WJ_0 \quad \text{for } \varepsilon = 1 \text{ or } \varepsilon = -1, \]

it was noticed by Phillips [5, Lemma 3.1] that the Krein extension of \( J_0 \) will also satisfy (3.4) in place of \( J_0 \). Our construction makes this obvious, for if (3.4) holds then \( W \) commutes with \( P \) and \( P_1 \), so \( J_1 \) satisfies (3.4) and therefore \( J^*_1 \) satisfies (3.4) and therefore \( (I-J_1J^*_1)^{1/2} \) commutes with \( W \) and finally the transformations \( L^* \) and \( L \) defined by (3.3) satisfy (3.4). It follows that the Krein extension of \( J_0 \) will also satisfy (3.4).

**References**


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