ADDENDA AND CORRIGENDA TO "ON FILIPPOV'S IMPLICIT FUNCTIONS LEMMA"

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1. The authors have had their attention called to a previously published note by C. Castaing [2] containing results that overlap considerably with Theorem 1.

2. A paper [3] by L. Cesari that appeared after proof-reading of [1] motivated the following strengthening and simplification of Theorem 2 of [1]; the connection between references [3] and [4] and the theorem will be explained after the proof.

**Theorem 2'**. Let $C^*$ be the union of countably many compact metrizable sets. For each $(x, t)$ in $\mathbb{R}^{n+1}$ let $C(x, t)$ be a subset of $C^*$ such that the set $M^*$ of all $(x, t, v)$ with $(x, t)$ in $\mathbb{R}^{n+1}$ and $v$ in $C(x, t)$ is a closed subset of $\mathbb{R}^{n+1} \times C^*$. Let $f_1, \ldots, f_n$ be continuous real-valued functions on $M^*$. Let $x: [a, b] \rightarrow \mathbb{R}$ be an absolutely continuous function such that for almost all $t$ in $[a, b]$, $x'(t)$ is contained in the convex cover of the image $f(x(t), t, C(x(t), t))$ of $C(x(t), t)$ in $\mathbb{R}^n$. Then there exist $n+1$ measurable functions $v_i: [a, b] \rightarrow C^*$ and $n+1$ measurable nonnegative functions $p_i: [a, b] \rightarrow \mathbb{R}$ such that for all $t$ in $[a, b]$, each $v_i(t)$ is in $C(x(t), t)$, and

\[
\sum_{i=1}^{n+1} p_i(t) = 1 \quad \text{and for almost all } t \in [a, b]
\]

\[
x''(t) = \sum_{i=1}^{n+1} p_i(t)f_i(x(t), t, v_i(t)).
\]

Let $W_{n+1}$ be the set of all $(n+1)$-tuples $(p_1, \ldots, p_{n+1})$ with all $p_j \geq 0$ and $\sum p_j = 1$. Then the set

\[ Q = (M^*)^{n+1} \times W_{n+1} \]

is the union of countably many metrizable compact sets. Let $k$ be the mapping from $Q$ into $\mathbb{R}^{n+3n+1}$ whose value at the point

\[ (x_1, t_1, v_1, \ldots, x_{n+1}, t_{n+1}, v_{n+1}, p_1, \ldots, p_{n+1}) \]

is given by

\[ k(s) = \sum_{j=1}^{n+1} p_j f_j(x_j, t_j, v_j) \quad (i = 1, \ldots, n), \]

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This is continuous on $Q$.

There is a subset $M$ of $[a, b]$ with measure $b-a$ such that for all $t$ in $M$, $x'(t)$ exists and is in the smallest convex set that contains $f(x(t), t, C(x(t), t))$. By a theorem of Carathéodory it therefore can be written as

$$x'(t) = \sum_{j=1}^{n+1} p_j f_i(x(t), t, v_i)$$

where $p$ is in $W_{n+1}$ and each $v_i$ is in $C(x(t), t)$. Therefore if in the expression (2) for $x'$ we choose each $t_j$ to be $t$ and each $x_j$ to be $x(t)$, we obtain

$$k^j(x) = x'(t) \quad (i = 1, \ldots, n),$$

$$k^{n+1}(x) = x'(t) \quad (j = 1, \ldots, n + 1; \quad i = 1, \ldots, n),$$

$$k^{n+2n+1}(x) = t \quad (j = 1, \ldots, n + 1).$$

We define $y: M \to R^{n+2n+1}$ by setting $y(t) = (x'(t), x(t), \ldots, x(t), t, \ldots, t)$, the dots denoting $(n+1)$-fold repetition. The preceding equations imply $y(M) \subseteq k(Q)$. Hence, by Theorem 1, there exists a measurable function $u: M \to Q$ such that

$$k(u(t)) = y(t) \quad (t \text{ in } M).$$

We denote $u(t)$ by

$$(x_1(t), t_1(t), v_1(t), \ldots, x_{n+1}(t), t_{n+1}(t), v_{n+1}(t), p_1(t), \ldots, p_{n+1}(t)).$$

Then (3) implies that for $i = 1, \ldots, n$ and $j = 1, \ldots, n+1$ we have

$$\sum_{j=1}^{n+1} p_j f_i(x_j(t), t_j(t), v_j(t)) = x'(t),$$

$$x'_j(t) = x'_j(t),$$

$$t_j(t) = t.$$

Substituting the last pair of equations in the one preceding then yields (1) for all $t$ in $M$, completing the proof.

The "chattering controls" of Gamkrelidze [4] are the functions $(p_1, \ldots, p_{n+1}, v_1, \ldots, v_{n+1})$ of (1); for generalized curves based on such controls, Gamkrelidze established the maximum principle. Amending the definition by allowing $v$ ($\geq n+1$) components in $p$ and
Cesari established the existence of an optimizing generalized curve. The generalized curves of Young and McShane replace (1) by

$$x'' = \int f'(x(t), t, v)\,\rho_t(dv)$$

with $\rho_t$ a probability measure on $C(x(t), t)$. By Theorem 2', if the convex hull of $f(x(t), t, C(x(t), t))$ is closed for all $t$, there is a chattering control in the sense of Gamkrelidze that yields the same trajectory; the different formulations are in effect interchangeable.

3. At the bottom of page 41, change $[0, \infty)$ to $(0, \infty)$.

4. The first line of Theorem 2 should read “If $C^*$ is the union of a countable set $K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots$.” However, the theorem is in fact correct even as misprinted, since this is a special case of Theorem 2'.

**References**


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