A REMARK ON REAL CHARACTERS OF COMPACT GROUPS

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1. In this note we shall extend a theorem of Frobenius and Schur [1] on real characters of finite groups. We want also to suggest a simple proof for this result using only basic linear algebra.

Let G be a compact group, and let M be a finite-dimensional irreducible representation module for G over the field C of complex numbers. We denote by χ the corresponding character of G. χ is called *realizable over* R (the field of real numbers), if there exists a RG-module N such that the CG-module $N\otimes_R C$ is isomorphic to M. Clearly, a necessary condition for χ to be realizable over R is that $\chi(g)$ be a real number for each g in G. If the latter is the case we call χ *real.* The problem of course is to decide which real characters are also realizable over R. We define an invariant $c=c(\chi)$ of χ by setting $c=1$ if χ is realizable over R, $c=-1$ if χ is real but not realizable over R, and $c=0$ if χ is not real. Then we have the following criterion:

$$c(\chi) = \int_G \chi(g^2)dg. \tag{1}$$

Here $\int$ denotes the normalized Haar integral of the compact group G (i.e. $\int_G dg=1$). Frobenius and Schur stated this theorem for the case of a finite group, where of course the integral on the right side of (1) is equal to the sum of all $\chi(g^2)$, g in G, divided by the order of G.

2. We consider the CG-module $M\otimes_C M$ to which the character χ^2 belongs. We have

$$M \otimes_C M = S(M) \oplus A(M) \tag{2}$$

where $S(M)$ (resp. $A(M)$) denotes the G-submodule of symmetric (resp. antisymmetric) tensors of $M\otimes_C M$. Hence it follows for the corresponding characters

$$\chi^2 = \chi_s + \chi_a. \tag{3}$$

There is a simple relation between the values of χ_s and χ_a, namely

$$\chi_s = \chi_a(g) + \chi(g^2) \tag{4}$$

(this is easily checked by taking a diagonal basis for the endomorphism of M corresponding to g).

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Making use now of the orthogonality relations for χ and its conjugate $\bar{\chi}$ (defined by $\bar{\chi}(g)=\overline{\chi(g)}=\chi(g^{-1})$) we see that $\int\chi^2(g)dg$ is equal to 1 if χ is real, and equal to 0 otherwise. This can also be expressed by saying that the tensors of $M\otimes_C M$ which are invariant under each g in G form a subspace of dimension 1 or 0 depending on whether χ is real or not. If we set $c'=\int\chi(g^2)dg=\int\chi_s(g)dg-\int\chi_a(g)dg$ we conclude from (2) that c' can only be 0, 1, or -1. Furthermore, $c'=0$ is equivalent to χ being not real and c' is equal to 1 if and only if there is a *symmetric* tensor in $M\otimes_C M$ not equal to zero and invariant under G. Therefore (1) will be a consequence of the theorem below:

THEOREM. *The following statements are equivalent:*

(i) *χ is realizable over R.*

(ii) *There exists a nonzero symmetric tensor in $M\otimes_C M$ which is invariant under G.*

3. From now on let χ be real. In order to prove the theorem just stated we use the following fact which is generally known under the name of "Weyl's trick": There exists on M a positive definite hermitian form h which is invariant under G, i.e., $h(gx, gy)=h(x, y)$ for all g in G and all x, y in M. (If h' is *any* positive definite hermitian form on M, define h by $h(x, y)=\int_G h'(gx, gy)dg$. Then h does the trick.)

For a moment we consider also the dual space M^0 of M endowed with its natural G-module structure: For every f in M^0 and every g in G the element of gf of M^0 is defined by $gf(x)=f(g^{-1}x)$ for all x in M. The corresponding character is $\bar{\chi}$. Since χ is real we must have $M\cong M^0$. By looking at the resulting isomorphism $M\otimes_C M\cong M^0\otimes_C M^0 \cong(M\otimes_C M)^0$ we see that condition (ii) above is equivalent to

(ii') *There exists a nonzero bilinear symmetric form b on M which is invariant under G, i.e. $b(gx, gy)=b(x, y)$ for all g in G and all x, y in M.*

We assume first that (ii') is satisfied. Since the form h is nondegenerate there is a map j: $M\rightarrow M$ such that

$$b(x, y) = h(x, jy) \qquad \text{for all } x, y \text{ in } M. \tag{5}$$

Furthermore, j is semilinear with respect to conjugation in C and $jgx=gjx$ for all g in G and all x in M. Then $j^2=j\circ j$ is an isomorphism of the CG-module M and it follows from Schur's Lemma that $j^2=c\,\mathrm{id}_M$ where c is a complex number and id_M the identity map on M. We assert that *c is a real number greater than zero.* To see this we set in (5) $y=jx$. Using the symmetry of b we get $h(x, j^2x)=h(jx, jx)$ or $\bar{c}h(x, x)=h(jx, jx)$. Our assertion follows from the fact that h is positive definite. Replacing now j by $c^{-1/2}j$ (where $c^{-1/2}$ is the reciprocal of a square root of c in R) we see that we can assume that $j^2 = \mathrm{id}_M$. But then it follows that for $N = \{x; jx = x\}$ we have $M = N \oplus iN = N \otimes_R C$, i.e. χ is realizable over R.

On the other hand, if χ is realizable over R, the map $j = \mathrm{id}_N \otimes \sigma$, where σ denotes conjugation in C, is a semilinear map of $N \otimes_R C$ with $j^2 = \mathrm{id}_M$ and such that $jgx = gjx$ for all g in G and all x in M. Then b defined by $b(x, y) = h(x, jy)$ is a nonzero bilinear form on M which is invariant under G, i.e. (ii') and hence (ii) is satisfied.

4. Let the group G now be finite. It would be interesting to know if anything similar to Frobenius and Schur's criterion can be said for quadratic extensions L/K other than C/R. There is one easy example which we want to mention here: Let K be a real field (i.e. -1 is not a sum of squares in K) and suppose further that K has the property that for every a in K either a or $-a$ is a square in K. It follows that K can be regarded as an ordered field whose positive elements are just the squares of K. There is one and only one quadratic extension L of K namely the field $L = K((-1)^{1/2})$. Then, if we replace R by K and C by L, the criterion of Frobenius and Schur remains valid also in this situation. The proof is completely analogous to the one given above in the case $K = R$ and $L = C$.

Reference

1. G. Frobenius and I. Schur, *Über die reellen Darstellungen der endlichen Gruppen*, S.-B. Preuss. Akad. Wiss. (1906), 186–208.

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