## SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is no other outlet.

# THE MINIMAX PRINCIPLE AND UNIQUENESS OF THE FRIEDRICHS EXTENSION

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If $S$ is a symmetric operator in a Hilbert space $\mathsf{H}$, and is bounded below, then the Friedrichs extension $T$ of $S$ (cf., for instance, Riesz and Nagy [2, pp. 327–333]) has the same lower bound as $S$. This property does not characterize $T$ uniquely except in special cases, but if $T$ has a purely discrete spectrum, the following holds:

THEOREM. *Let $T'$ be a selfadjoint extension and $T$ the Friedrichs extension of $S$. Assume that both have compact resolvent, and let $(\lambda_n')_{n\in\mathsf{N}}$ resp. $(\lambda_n)_{n\in\mathsf{N}}$ be the eigenvalues for $T'$ and $T$, nondecreasingly ordered, and appearing with appropriate multiplicities. If $\lambda_n' = \lambda_n$ for some $n$, then $T'$ and $T$ have a common eigenvector corresponding to this eigenvalue. If $\lambda_n' = \lambda_n$ for all $n$, then $T' = T$.*

The first assertion of this theorem is a simple consequence of the following slight elaboration of the well-known minimax principle (the terminology being that of Kato [1]):

*Let $\mathfrak{t}$ be the closed quadratic form associated with the selfadjoint semibounded operator $T$. Let $\mathsf{M}$ be a subspace of $\mathsf{H}$ of codimension $n-1$, and assume that*

$$\min\{\mathfrak{t}[v,v]\,|\,v\in \mathsf{M}\cap D(\mathfrak{t}), \|v\| = 1\} = \lambda_n.$$

*Then $\mathsf{M}$ contains an eigenvector for $T$ corresponding to the eigenvalue $\lambda_n$.*

The second assertion follows from the first when it is observed that a subspace invariant under both $T$ and $T'$ reduces both of these as well as the corresponding quadratic forms.

Observe in conclusion that if $\lambda_n' = \lambda_n$ for all but a finite number of $n$'s, then $T' = T$, for then the quadratic forms have the same domain, and hence are identical.

ADDED IN PROOF. K. Jörgens has called the author's attention to the reference Berkowitz [3], and pointed out that the theorem remains true with $\lambda_k$ defined as in that paper provided $\lambda_1, \lambda_2, \cdots, \lambda_n$

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are eigenvalues for $T$. Jörgens also suggested that it be mentioned that one may have $\lambda_n' = \lambda_n$ for infinitely many $n$, but $T' \neq T$ even if $S = -d^2/dx^2$ on $\mathcal{C}_0^\infty(0, 1)$.

## References


1. T. Kato, *Perturbation theory for linear operators*, Springer, New York, 1966.

2. F. Riesz and Béla Sz.-Nagy, *Leçons d'analyse fonctionnelle*, Académiai Kiadó, Budapest, 1952.

3. J. Berkowitz, *On the discreteness of spectra of singular Sturm-Liouville problems.* Comm. Pure Appl. Math. 12 (1959), 523-542.


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