SOME LARGE O NONLINEAR TAUBERIAN THEOREMS[1]

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The following "large-O" nonlinear Tauberian Theorems are extensions of the "small-O" theorems of Boas [1], Karamata [2] and Pollard [3]. Motivation for these theorems comes from the problem of n-bodies in celestial mechanics. Here bounds on the behavior of the self-potential, U, are sought starting from the knowledge that $\int_0^t U(s)ds = O(t^a)$ ($t \to 0+$, or $t \to \infty$) and $|\dot{U}| \leqq CU^{5/2}$. Here C and a are constants.

In the following the symbols f, g and h represent functions which are of the class C^2 on $(0, \infty)$ and ω, ϕ represent positive continuous functions.

$f(t) = O(t)$ has its standard meaning; $f(t) \leqq O(t)$ means that $f(t)/t$ is bounded above (but not necessarily below) as t approaches its specified limit.

THEOREM 1. *If*

$$g(t) = O(t) \quad \textit{as } t \to 0+, \qquad \textit{and} \qquad g''(t) \leqq \omega(g'(t))O(\phi(t)),$$

where $\phi(t)$ *is integrable, then*

$$g'(t) = O(1) \qquad \textit{as } t \to 0+.$$

Note that this includes $\phi(t) = t^\alpha$ where $\alpha > -1$.

PROOF. Since $g(t) = O(t)$, a $B > 0$ can be found such that for all positive t less than some value, $|g(t)| \leqq Bt$. Define $A = \{t: |g'(t)| \leqq B\}$.

Claim. A is nonempty and has zero as a limit point. If this were not so then there would exist a neighborhood of zero $(0, t)$ such that $|g'(t)| > B$. The continuity of g' implies that it has one sign in this neighborhood. Thus $|g(t)| = \int_0^t |g'(u)| du > Bt$, a contradiction.

To simplify the formulas, set $y = g'(t)$. Then $dy/dt = g''(t)$ and the hypothesis implies that for positive t less than some value

$$dy/dt \leqq C\omega(y)\phi(t).$$

As $\Phi(t) = \int_{0+}^t \phi(u)du$ is a positive increasing function, it follows that

$$W(y_1) - W(y) = \int_y^{y_1} \frac{dy}{\omega(y)} \leqq C\int_t^{t_1} \phi(s)ds < C\Phi(t_2) \tag{1}$$

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where W is defined by (1) and $t_2 \geqq t_1$.

Note that W is an increasing function ($\omega(y)>0$).

Claim. y is eventually bounded above by $2B$. Assume this to be false. Then an $\epsilon>0$ can be determined such that $W(2B)-W(B)>\epsilon$. Choose t_2 small enough so that $C\Phi(t_2)<\epsilon$. Now assume t_1 is such that $y_1 \geqq 2B$. By properties of set A and the choice of y_1, there is a t such that $0<t<t_1$ and $y(t)=B$. Substituting the above into (1) yields $\epsilon > C\Phi(t_2) > W(y_1)-W(B) \geqq W(2B)-W(B)>\epsilon$. From this contradiction it follows that y is bounded above. A similar proof is employed to show that y is bounded below and the proof is completed.

It is clear that the corresponding theorem for $t\to\infty$ holds also. The proof is given most simply by modifying the reasoning rather than a change in variable.

THEOREM 2. *If*

$$g(t) = O(t) \quad \text{as } t \to \infty, \qquad \text{and} \qquad g''(t) \leqq \omega(g'(t))O(\phi(t)),$$

where $\int_t^\infty \phi(s)\,ds$ *is a convergent integral, then*

$$g'(t) = O(1) \qquad \text{as } t \to \infty.$$

The above two theorems can be sharpened to include the case $\phi(t)=t^{-1}$. This will be stated and proved here only when $t\to\infty$. The corresponding statement for $t\to 0+$ is the same.

THEOREM 3. *If* $|g(t)| \leqq Ct$, $t\to\infty$, *and* $g''(t) \leqq \omega(g'(t))t^{-1}$, *then* g' *is bounded above if either*:

(1) $\int_a^x du/\omega(u)$ *is unbounded as* $x\to\infty$,

(2) $\lim\, (x^2-1)\,[1/\omega(Cx)-n/\omega(nCx)]>2/C$ *as* $x\to\infty$, *and* n *is some constant* >1. g' *is bounded below if either* (1) *or* (2) *is true for* $x\to-\infty$.

PROOF. It will be shown here that g' is bounded above. The proof that it is bounded below is obtained in a similar manner.

Claim. If $g' \geqq MC$, where $M>1$, it is so at most on an interval $(t, \lambda t)$ where $\lambda \leqq (M+1)/(M-1)$.

By the hypothesis and the assumption on g';

$$(\lambda+1)Ct \geqq g(\lambda t) - g(t) = \int_t^{\lambda t} g'(s)ds \geqq MCt(\lambda-1),$$

or $\lambda \leqq (M+1)/(M-1)$.

For notational purposes set $y=g'(t)$. Assign a fixed value to M and consider an interval $[t_1, t]$ on which $y \geqq MC$ and $y(t_1)=MC$. (If no such interval exists, then y is bounded above by MC.) By the conditions imposed upon $t/t_1=\lambda$ and the second derivative of g,

$$(2) \qquad W(y) - W(MC) = \int_{MC}^{y} \frac{ds}{\omega(s)} \leqq \ln(t/t_1) \leqq \ln\left(\frac{M+1}{M-1}\right).$$

If W is an unbounded (increasing by virtue of $\omega > 0$) function, then it follows immediately from (2) that y is bounded above.

If W is bounded above, a slight modification in terms of extra conditions is necessary to apply the above technique. A sufficient condition would be if, for some $n > 1$, an $M < \infty$ could be found such that $W(nCM) - W(CM) > \ln((M+1/M-1))$. y would then be bounded above by nCM.

If the limit exists, such an M can be found if for some $n > 1$,

$$\lim \frac{W(nCx) - W(Cx)}{\ln(x+1)/(x-1)} > 1 \qquad \text{as } x \to \infty,$$

or by L'Hospital's rule, if

$$\lim (x^2 - 1)(1/\omega(C_y) - n/\omega(nC_y)) > 2/C \qquad \text{as } x \to \infty.$$

Hence under the above conditions, g' is bounded above.

If ω is bounded, then we have as a special case of Theorem 3 a well-known linear Tauberian Theorem: $g(t) = O(t)$, $g'' \leqq Ct^{-1} \Rightarrow g' = O(1)$ as $t \to \infty$ (or $t \to 0+$).

The above theorems can be rewritten in a form which includes the motivating problem at the beginning of this note.

THEOREM 4. *If*

$$h(t) = O(t^\alpha), \qquad t \to \infty, \qquad \alpha \neq 0,$$
$$|h''(t)| \leqq B|h'(t)|^\delta t^\beta,$$

and $h'(t)$ is of one sign after some value of t, then $h'(t) = O(t^{\alpha-1})$ if any of the following hold.

(i) $\eta = (\delta - 1)(\alpha - 1) + \beta + 1 < 0$.

(ii) $\delta < 2$ *and* $\eta \leqq 0$.

(iii) $|h(t)| \leqq Ct^\alpha$, $\delta = 2$, $2BC < \alpha^2$ *and* $\eta \leqq 0$.

PROOF. It can be assumed without loss of generality that $h' > 0$. Define

$$g(t^\alpha) = h(t) = O(t^\alpha), \qquad \alpha g'(t^\alpha) t^{\alpha-1} = h'(t)$$

and

$$g''(t^\alpha)\alpha^2 t^{2\alpha-2} + g'(t^\alpha)\alpha(\alpha - 1)t^{\alpha-2} = h''(t).$$

Note that $g'(x)$ is of one sign. If $\alpha = 1$, the above is reduced to the conditions of Theorems 2 and 3 and this theorem follows.

$\alpha > 1$. Substituting the above for h in the second derivative condition and noting that the term containing g' is nonnegative, it follows that

$$\alpha^2 t^\alpha g''(t^\alpha) \leqq B \left| g'(t^\alpha) \right|^\delta t^{(\delta-1)(\alpha-1)+\beta+1}.$$

By substituting $x = t^\alpha$ and applying the conditions of Theorems 2 and 3, this theorem follows.

If $0 < \alpha < 1$ the proof is the same except the term involving g' is now negative, so the other inequality available from the second derivative condition is employed. For negative α, $x \to 0$ and Theorems 1 and 3 are used. Notice that the absolute value of h'' is not necessary, only the appropriate inequality depending on the value of α and the sign of $h'(t)$.

If $\delta = 0$, Theorem 4 has as a special case a well-known linear Tauberian Theorem. Note the interesting returns if $\delta = 1$. All that is necessary for all $\alpha \neq 0$ is $\beta \leqq -1$.

Following the reasoning of Pollard [3] Theorem 4 can be extended to the case $\alpha = \infty$ in the following form.

THEOREM 5. *If*

$$h(t) = O(e^t), \qquad t \to \infty,$$

and

$$|h''(t)| \leqq B \left| h'(t) \right|^\delta, \qquad \delta < 2,$$

then

$$h'(t) = O(e^t), \qquad t \to \infty.$$

PROOF. The function $g(e^t) = h(t)$ satisfies the conditions of Theorem 3 where $\omega(y) = |y|^\delta + |y| + 1$.

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