REGULARITY OF NET SUMMABILITY TRANSFORMS
ON CERTAIN LINEAR TOPOLOGICAL SPACES

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The famous Silverman-Toeplitz theorem concerning the regularity of a matrix transform $A$ with complex entries has undergone much study and many generalizations. Among these have been contributions by Kojima [5], Fraleigh [3], Adams [1], and Agnew [2] concerning the summability of multiple sequences. Melvin-Melvin [6] examined the case where each entry in $A$ is a bounded linear operator on a Banach space. Recently Ramanujan [7] extended this idea to two types of linear topological spaces; namely, Fréchet spaces and locally bounded spaces. It is the purpose of this paper to generalize the results of Ramanujan by allowing the "rows" and "columns" of $A$ to be nets of linear continuous transformations and to investigate conditions under which $A$ will transform certain convergent nets into convergent nets.

We shall require the following notation and assumptions:

$X, Y$: linear spaces over the complex numbers;
$\mathfrak{K}$: a collection of seminorms (see [8] for definition) on $X$ which separates points of $X$ (i.e., if $x \in X - \{0\}$, then there is an $N \in \mathfrak{K}$ such that $N(x) > 0$);
$\mathfrak{M}$: a collection of seminorms on $Y$ that separates points;
$\mathfrak{G}$: the locally convex, Hausdorff topology on $X$ generated by $\mathfrak{K}$;
$\mathfrak{G}$: the locally convex, Hausdorff topology on $Y$ generated by $\mathfrak{M}$;
$\{D, \leq\}$: a directed set with finite initial segments (i.e., if $d \in D$, then the set $\{e | e \in D \text{ and } e \leq d\}$ is finite);
$C$: the collection of all bounded convergent nets from $D$ to $X$.
In this context, a net will be said to be convergent provided it has a limit. For a general discussion of nets, see [4]. If $f \in C$ and $N \in \mathfrak{K}$, then let $N^*(f) = \sup \{N(f(d)) | d \in D\}$.
$\mathfrak{K}^*$: the set of all $N^*$, for $N \in \mathfrak{K}$;
$\mathfrak{G}^*$: the locally convex, Hausdorff topology on $C$ generated by $\mathfrak{K}^*$.

We shall assume, for the remainder of the paper, that the linear topological space $\{C, \mathfrak{G}^*\}$ is barrelled (see [8] for definition). This

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condition is necessary and sufficient (so long as $Y$ is nondegenerate) to insure that the following version of the Uniform Boundedness Principle holds. For a proof and related results, see Chapter IV of [8]; and, particularly, Theorem 3 on page 69.

**Theorem 1.** If $\mathcal{M}$ is a collection of linear continuous operators from $\{C, \mathfrak{F}\}$ to $\{Y, \mathfrak{G}\}$ with the property

$$
\text{if } f \in C \text{ and } M \in \mathcal{M}, \text{ then } \{M(T(f)) \mid T \in \mathcal{M}\} \text{ is bounded;}
$$

then, for each $M \in \mathcal{M}$, there is a finite subset $\{N_i\}_{i=1}^{\mathfrak{N}}$ of $\mathfrak{N}$ and a positive number $K$ such that

$$
\text{if } f \in C \text{ and } T \in \mathcal{M}, \text{ then } M(T(f)) \leq K \sup_{i \leq \mathfrak{N}} N_i(f).
$$

Observe that, if $\mathfrak{N}$ is countable and $\{X, \mathfrak{F}\}$ is complete, then $\{C, \mathfrak{F}\}$ is second category and thus barrelled. That is, if $\{X, \mathfrak{F}\}$ is a Fréchet space then $\{C, \mathfrak{F}\}$ is barrelled.

Let $A$ be a function from $D \times D$ into the set of linear continuous functions from $\{X, \mathfrak{F}\}$ to $\{Y, \mathfrak{G}\}$ (denoted hereafter by $L(X, Y)$). The $D \times D$ matrix $A$ is said to be convergence preserving provided that, if $f \in C$ and $d \in D$, then the net $h(e) = \sum_{e' \in \mathfrak{N}} A(d, e')f(e')$ is bounded and convergent in $\{Y, \mathfrak{G}\}$; and the net $A(f)$ defined, for $d \in D$, by $A(f)(d) = \lim_{e} h(e)$ is also bounded and convergent in $\{Y, \mathfrak{G}\}$. Further, if $L \in L(X, Y)$, then $A$ is said to be $L$-regular provided $A$ is convergence preserving and, if $f \in C$, then $A(f)$ has limit $L(\lim_{d} f(d))$.

The main result of this paper is the following Toeplitz theorem which generalizes Theorem 1 of [7].

**Theorem 2.** Suppose $\{Y, \mathfrak{G}\}$ is complete—in the sense that every Cauchy net from $D$ to $Y$ is a convergent net. The function $A$ is convergence preserving if, and only if, the following four statements are true:

1. For each $M \in \mathcal{M}$, there is a finite subset $\{N_i\}_{i=1}^{\mathfrak{N}}$ of $\mathfrak{N}$ and a positive number $K$ such that, if $f \in C$ and $(d, e) \in D \times D$, then

$$
M\left( \sum_{e' \in \mathfrak{N}} A(d, e')f(e') \right) \leq K \sup_{i \leq \mathfrak{N}} N_i(f).
$$

2. If $f \in C$ and $d \in D$, then the net $h(e) = \sum_{e' \in \mathfrak{N}} A(d, e')f(e')$ is convergent in $\{Y, \mathfrak{G}\}$.

3. If $f \in C$ with limit 0 and $d \in D$ then the net defined, for $d_0 \in D$, by

$$
g(d_0) = \lim_{e} \sum_{e' \in \mathfrak{N}, \delta \leq e'} A(d_0, e')f(e'),
$$

is convergent in $\{Y, \mathfrak{G}\}$.
(4) If $x \in X$, then the net defined, for $d \in D$, by
\[ k(d) = \lim \sum_{e' \leq e} A(d, e')(x), \]
is convergent in $\{Y, g\}$.

**Proof.** Suppose that $A$ is convergence preserving. For each pair $(d, e)$ in $D \times D$, let $T_{(d, e)}$ be the linear function from $C$ to $Y$ defined, for $f \in C$, by $T_{(d, e)}(f) = \sum_{e' \leq e} A(d, e')f(e')$. Since $D$ has finite initial segments and $A$ has entries in $\mathcal{L}(X, Y)$, it is easily seen that $T_{(d, e)}$ is continuous from $\{C, \mathcal{F}_e\}$ to $\{Y, g\}$. Furthermore, if $d \in D$ and $f \in C$, then the net $h(e) = T_{(d, e)}(f)$ is bounded in $\{Y, g\}$. Suppose $M \in \mathfrak{M}$ and $d \in D$. By Theorem 1, there is a finite subset $\{N_i(f)\}_{i=1}^n$ in $\mathfrak{M}$ and a positive number $K$ such that, if $e \in D$ and $f \in C$, then
\[ M(T_{(d, e)}(f)) \leq K \sup_{1 \leq i \leq n} N_i(f). \]
Thus, if $f \in C$,
\[ M\left(\lim_\epsilon T_{(d, e)}(f)\right) \leq K \sup_{1 \leq i \leq n} N_i(f); \]
and the linear function defined, for $f \in C$, by $T_d(f) = \lim_\epsilon T_{(d, e)}(f)$ is continuous from $\{C, \mathcal{F}_e\}$ to $\{Y, g\}$. Finally, if $f \in C$, then the net $k(d) = T_d(f)$ is bounded in $\{Y, g\}$ and a reapplication of Theorem 1 shows that statement (1) holds. Statement (2) is immediate from the definition. Concerning statement (3), suppose $d \in D$ and $f \in C$. If $e \in D$, let
\[ f'(e) = f(e) \quad \text{if } d \neq e, \]
\[ = 0 \quad \text{if } d < e. \]
Then $f' \in C$ has limit 0; and, if $(d_0, e) \in D \times D$,
\[ \sum_{e' \leq e} A(d_0, e')f'(e') = \sum_{e' \leq e' \leq k} A(d_0, e')f(e'). \]
Statement (3) now follows. Finally, let $x \in X$ and define, for $d \in D$, $f(d) = x$. Thus $f \in C$ and statement (4) follows from statement (2) and the definition.

To prove the converse, suppose that statements (1), (2), (3) and (4) hold; $M \in \mathfrak{M}$, $\epsilon > 0$, and $f \in C$. Let $\{N_i(f)\}_{i=1}^n$ be a finite subset of $\mathfrak{M}$ and $K$ a positive number with the properties of statement (1). Let $d_0 \in D$ be such that, if $d \in D$ and $d_0 \leq d$, then $N_i(f(d) - \lim f) < \epsilon/4K$ for $i = 1, 2, \ldots, n$. Secondly, let $d_1 \in D$ be such that, if $d \in D$ and $d_1 \leq d$, then
(a) \[ M \left( \lim_{e' \in E} \sum_{e' \in E} A(d, e')(\lim f) - \lim_{e' \in E} \sum_{e' \in E} A(d_1, e')(\lim f) \right) < \frac{\varepsilon}{4}. \]

and

(b) \[ M \left( \lim_{e' \in E} \sum_{e' \in E} A(d, e')(f(e') - \lim f) - \lim_{e' \in E} \sum_{e' \in E} A(d_1, e')(f(e') - \lim f) \right) < \frac{\varepsilon}{4}. \]

If \( d \in D \) and \( d_1 \leq d \), then

\[ M \left( \lim_{e' \in E} \sum_{e' \in E} A(d, e')f(e') - \lim_{e' \in E} \sum_{e' \in E} A(d_1, e')f(e') \right) < \varepsilon. \]

Thus, since \( \{ Y, g \} \) is net complete, \( A \) is convergence preserving and Theorem 2 is proved.

An analogous argument will yield a proof of the following theorem where \( L \subseteq \mathcal{E}(X, Y) \).

**Theorem 3.** The function \( A \) is \( L \)-regular if, and only if, statements
(1) and (2) of Theorem 2 hold and the following two statements also hold:

(3') If \( f \in C \) with limit 0 and \( d \in D \) then the net defined, for \( d_0 \in D \), by

\[ g(d_0) = \lim_{e' \in E} \sum_{e' \in E} A(d_0, e')f(e'), \]

is convergent to 0 in \( \{ Y, g \} \).

(4') If \( x \in X \), then the net defined, for \( d \in D \), by

\[ k(d) = \lim_{e' \in E} \sum_{e' \in E} A(d, e')(x), \]

is convergent to \( \lambda(x) \) in \( \{ Y, g \} \).

It should be noted that in Theorem 3 we need not require that
\( \{ Y, g \} \) be net complete.

We see that Theorems 2 and 3 are extensions of Theorems 1 and 2 of [7], as well as the usual Silverman-Toeplitz theorems. However, there is one application which is important in the study of functions of several complex variables that is included in the present theory but not included in [7]. In particular, suppose that each of \( \{ X, \mathcal{F} \} \) and \( \{ Y, g \} \) is the complex plane with the usual topology, \( k \) is a positive integer, and \( D \) is the set of \( k \)-tuples of positive integers where, if each of \( n \) and \( m \) is in \( D \), then \( n \leq m \) provided that, for \( i = 1, 2, \ldots, k \), \( n(i) < m(i) \). In this case, letting \( I \) denote the identity function on \( X \),
we have the following theorem (of course, in this setting, \( A \) is simply a \( D \times D \) matrix of complex numbers).

**Theorem 4.** The function \( A \) is \( I \)-regular if, and only if, the following three statements are true:

1. There is a positive number \( K \) such that, if \((d, e) \in D \times D\), then \( \sum_{e' \in e} |A(d, e')| \leq K \).
2. If \( d \in D \), then the complex number net defined, for \( d_0 \in D \), by
   \[
   g(d_0) = \lim_{e' \in e; d_0 \notin e'} \sum_{e' \in e} |A(d_0, e')|
   \]
   is convergent to 0.
3. The net defined, for \( d \in D \), by
   \[
   k(d) = \lim_{e' \in e} \sum_{e' \in e} A(d, e')
   \]
   has limit 1.

**Proof.** In this context, statements (1) and (2) of Theorem 2 are equivalent to statement (1). Also statement (4') of Theorem 3 is equivalent to statement (3). As statement (2) implies statement (3') of Theorem 3, all that remains is to show that if \( A \) is \( I \)-regular, then statement (2) holds. Hence, suppose that \( A \) is \( I \)-regular, \( d \in D \); and for each \( d_0 \in D \), let

\[
 g(d_0) = \lim_{e' \in e; d_0 \notin e'} \sum_{e' \in e} |A(d_0, e')|.
\]

Suppose further that \( g \) does not have limit 0, and let \( \epsilon > 0 \) be such that, if \( e \in D \), there is an \( e' > e \) with the property that \( g(e') > \epsilon \). Let \( S = \{ d' \mid d' \in D, d \neq d' \}, e_i \in D \) such that \( e_i > 1 \) (the constant 1 member of \( D \)) and \( K_1 \) a finite subset of \( S \) such that

\[
 \sum_{e \in S} |A(e_1, e)| - \sum_{e \in K_1} |A(e_1, e)| < \frac{\epsilon}{4}.
\]

Suppose \( p \) is a positive integer and disjoint finite subsets \( K_1, K_2, \ldots, K_p \) of \( S \) have been chosen, along with elements \( e_1, e_2, \ldots, e_p \) of \( D \). Using the fact that if \( e \in D \), then the net \( A(\cdot, e) \) has limit 0; let \( e_{p+1} \in D \) be such that

\[
 e_p < e_{p+1}, \quad \sum_{e \in K_1} |A(e_{p+1}, e)| < \frac{\epsilon}{4}, \quad \text{and} \quad g(e_{p+1}) > \epsilon.
\]

Let \( K_{p+1} \) be a finite subset of \( S \) such that \( K_{p+1} \cap \cup_{i=1}^{p} K_i = \emptyset \) and
Define the member $f$ of $C$ as follows: if $e \in D$, then

$$f(e) = \frac{|A(e_n, e)|}{A(e_n, e)} \quad \text{if} \quad e \in K_n \quad \text{and} \quad A(e_n, e) \neq 0,$$

$$= 0, \quad \text{otherwise.}$$

If $p$ is a positive integer, then

$$\lim_{e \to e_p} \sum_{e' \in S} |A(e_p, e')f(e')| \geq \sum_{e' \in K_p} |A(e_p, e')| - 2 \sum_{e' \in S - K_p} |A(e_p, e')|$$

$$> \epsilon - \epsilon/2 = \epsilon/2.$$

Since the sequence $e$ is cofinal in $D$, we see that $A(f)$ does not have limit 0. This contradiction establishes Theorem 4.

**Bibliography**


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