COLLAPSING A TRIANGULATION OF A "KNOTTED" CELL[1]

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1. **Introduction.** An interesting problem in combinatorial topology is to discover which triangulations of the 3-cell are simplicially collapsible. Chillingworth [2] has shown that if $|L| \cong B^3$ is linearly embedded as a convex subset of E^3, then $L \searrow^s 0$. On the other hand Goodrick [3] has shown that if L^1, the 1-skeleton of L, contains an n-bridge knot $(n>2)$ which except for one 1-simplex lies in ∂L, then L is not simplicially collapsible. This result was announced earlier by Bing [1].

An n-bridge knot is one which can be realized entirely on the surface of a standard 3-ball except for n straight line-segments in the interior which connect points in the boundary, and which cannot be realized with fewer than n such "bridges." Two-bridge knots have been completely classified by Schubert [5], and it is known that the sum of k 2-bridge knots is a $(k+1)$-bridge knot.

The purpose of this note is to show that if L^1 contains any 2-bridge knot which except for a single 1-simplex lies in ∂L, then L may be simplicially collapsible. In other words, the results of Bing and Goodrick cannot be extended to the case $n=2$. This fact is expressed by Theorem A. Theorem B is a partial extension of this result to n-bridge knots.

THEOREM A. *If K' is any 2-bridge knot, there exists a triangulated 3-cell L such that*

(a) *L^1 contains a knot K which is in ∂L except for one 1-simplex which is in the interior of L.*

(b) *K is of the same 2-bridge knot type as K'.*

(c) *L is simplicially collapsible ($L \searrow^s 0$).*

THEOREM B. *For any n there exists an n-bridge knot K_n and a triangulated 3-cell L_n such that*

(a) *L_n^1 contains K_n which is in ∂L_n except for $(n-1)$ 1-simplexes which are in the interior of L_n.*

(b) *L_n is simplicially collapsible.*

A different proof of Theorem A is given by Lickorish and Martin [4]. An unpublished proof of Theorem A has also been given by Goodrick.

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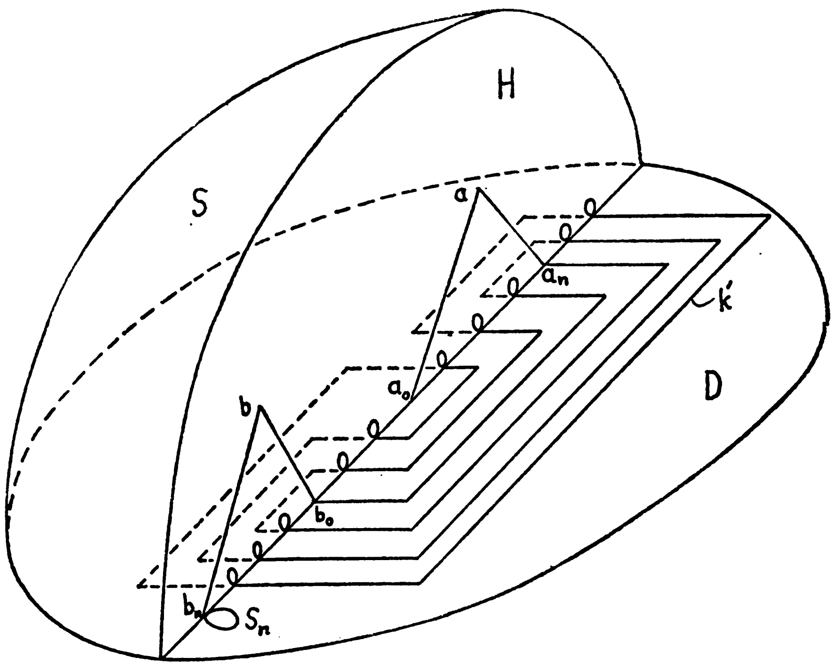

2. **The normal form of a 2-bridge knot.** Schubert [5] describes a normal form for 2-bridge knots that is completely determined by an ordered pair (α, β) of integers and gives necessary and sufficient conditions that two normal forms represent the same knot. We describe the normal form here and start to prove the theorems.

Let K' be an oriented 2-bridge knot. Then K' is ambient isotopic in E^3 to a knot (the normal form), which we still denote by K', that can be described as follows. Let D be the unit disc in the xy-plane of E^3, H the upper half ($z \geqq 0$) of the unit disc in the xz-plane and l the intersection of H and D, i.e. the interval $[-1, 1]$ of the x-axis. Let a_n, a_0, b_0, b_n be points of $(-1, 1)$ in that order and a and b points of H such that the solid triangles $\Delta_a = a_n a a_0$ and $\Delta_b = b_n b b_0$ are disjoint. The directed arcs $a_n a a_0$ and $b_n b b_0$ are the bridges of K', the direction being that of the orientation of K'. The rest of K' consists of two arcs, each of which lies in $\text{int}(D)$. One arc starts at a_0, undercrosses $b_n b b_0$ and $a_n a a_0$ alternately (beginning with $b_n b b_0$) and ends at b_n. The other starts at b_0 and after alternate undercrossings ends at a_n. In all, K' undercrosses $a_n a a_0$ at the points $a_1, \cdots, a_{n-1}$ which are selected on l in that order between a_0 and a_n and K' undercrosses $b_n b b_0$ at points $b_1, \cdots, b_{n-1}$ of l, selected in that order between b_0 and b_n. It may be required (and we do require) that K' does not meet the interval (b_n, ∞) of the x-axis.

Then n is Schubert's number α. Given such a knot, Schubert's number β is defined as follows. Recalling that K' is oriented and that the positive directions on the bridges are $a_n a a_0$ and $b_n b b_0$, attach to each point $a_i(b_i)$ the number i if the undercrossing is from right to left, $-i$ otherwise. Then if A is any arc of K' running in the positive direction from one undercrossing to the next, assign to A the number attached to its positive end minus the number attached to its negative end. The congruence class mod 2α of this number turns out to be independent of the arc A. Schubert's number β is the member of this class in the interval $(-\alpha, \alpha)$.

Conversely, given any pair of integers (α, β) relatively prime and with $-\alpha<\beta<\alpha$ one can construct a knot or link of type (α, β) satisfying the description above. For example, the arc starting at a_0 first undercrosses $b_n b b_0$ at b_β from right to left if $\beta>0$. If α is even one gets two linked circles, if odd a 2-bridge knot. If α is odd and β is even then the two knots (α, β) and $(\alpha, \beta-\alpha)$ (or $(\alpha, \beta+\alpha)$) are the same, and in this case $\beta\pm\alpha$ is odd. Thus any 2-bridge knot is represented by a pair (α, β) in which the integers are relatively prime and both odd.

REMARK 1. If we remove the directed arc of K' from a_0 to b_n and replace it by an unknotted arc from a_0 to b_n that lies below the xy-plane, then the resulting knot is ambient isotopic in E^3 to K'. Let k' denote the directed arc in K' from b_n to a_0.

REMARK 2. It turns out that the simple closed curve made up of the interval $[a_{|n-\beta-1|}, a_{|n-\beta+1|}]$ of l plus the arcs on k' from $a_{|n-\beta-1|}$ to b_{n-1} and b_{n-1} to $a_{|n-\beta+1|}$ bounds a disc B in D whose interior contains the rest of $k'\cap D$. The subscript $|k|$ here means the absolute value of the representative modulo 2α of k which is in the interval $(-\alpha, \alpha)$.

The figure shows the segment k' of the knot $(7, -5)$ together with some indications of the construction described below. The interval which completes the boundary of the disc B in this example is $[a_1, a_3]$. To complete the knot $(7, -5)$ one must connect a_0 to b_n (in this case b_7) either by winding an arc along the obvious channel in the disc D or by any unknotted arc below the disc D.

3. **Construction of the 3-cell** L. Add to $D\cup H$ the set $S = \{(x, y, z): x^2+y^2+z^2=1,\ z\geq 0,\ y\leq 0\}$, which is one quarter of the unit 2-sphere. Add to $D\cup H\cup S$ a 2-disc T which has the form of a long thin tube pinched to a point at one end. The other end, $\partial T=S_n$ is a small circle in D tangent to l at b_n, which bounds a small *open* disc $D_n\subset D-l$. We remove D_n from the construction. The tube T is constructed to be tangent to k', starting from b_n, and pinches to a point at a. Except for those parts of T that are tangent to $b_n b b_0$ and $a_n a$, T

lies on top of D and pierces H in small circles at every point a_i and b_j where k' pierces H, namely where i is odd and j is even. (We must put S_n on the correct side of l so that T does not pierce H at b_0.) These small circles bound open discs in H that are tangent to l at these a_i and b_j. The unions of these circles (discs) in $\triangle_a$ and $\triangle_b$ we denote by S_a (D_a) and S_b (D_b). The union of these circles (discs) in H but not in $\triangle_a$ or $\triangle_b$ we denote by $S_l(D_l)$. We remove from the construction the set $D_a \cup D_b \cup D_l$, i.e. we take away all those small open discs that block the tube T. Finally we remove $(\triangle_a - D_a)^0$. The resulting set $(D \cup H \cup S \cup T - D_n - D_a - D_b - D_l - (\triangle_a - D_a)^0)$, triangulated as a subcomplex of E^3, we denote by M. We require that the interval aa_0 be a 1-simplex of the triangulation.

To complete the construction of the 3-cell, L, take a regular neighborhood N of M in E^3. This is also triangulated so that aa_0 is a 1-simplex of N. There is an open 3-simplex s_1^3 in N with one vertex at a_0 and with a 2-face s_1^2 in the bottom ($z<0$) surface of N. There is also an open 3-simplex s_2^3 with one vertex at a and with a 2-face s_2^2 that lies in ∂N inside the narrowed tube. Put $L = N - (s_1^3 \cup s_1^2 \cup s_2^3 \cup s_2^2)$. In §4, we prove that L is a 3-cell.

To get a knot K that is embedded in L^1 in the prescribed way, take a path in L^1 from a_0 across the bottom of L to the opening D_n, which, though diminished by taking the regular neighborhood, still forms the entrance to the knotted hole. At the far end of the hole, due to the removal of $s_2^3 \cup s_2^2$, we find $a \in \partial L$. Continue the path in $\partial L \cap L^1$ through the tube to a and then along aa_0 to a_0. The 1-simplex aa_0 is then the single interior 1-simplex of the simple closed path we have constructed. This path is a knot K ambient isotopic to K' in E^3. (Remark 1 at end of §2.)

4. Proof of Theorem A.

LEMMA. *M is simplicially collapsible.*

PROOF. We carry out the collapse in steps. The symbol $l(x, y)$ denotes the interval (x, y) of l. Note that aa_0 is a free edge of M.

1. Collapse $H - (\triangle_a^0 \cup l(a_0, a_n) \cup \triangle_b^0 \cup l(b_0, b_n) \cup D_l)$ from aa_0 to

$$[\partial H - l(a_0, a_n) - l(b_0, b_n)] \cup S_l \cup aa_n \cup b_n b b_0.$$

We cannot collapse across $b_n b b_0$ or aa_n, since the tube is attached to these arcs.

2. Collapse S from $\partial H - l$ to $\partial S - (\partial H - l)$.
3. Collapse $D - (B^0 \cup D_n)$ from ∂D to $\partial B \cup l(b_{n-1}, b_n) \cup S_n$.

At this stage, $l(b_{n-1}, b_n)$ is a free edge of $\triangle_b$.

4. Collapse $\triangle_b - D_b$ from $l(b_{n-1}, b_n)$ to $b_n b b_0 \cup l(b_0, b_{n-1}) \cup S_b$.

We now have nothing left but $B \cup T$—i.e. in Collapses 1 through 4, $M \searrow^s B \cup T$. We see that S_n is a free edge of T at this stage.

5. Collapse T from S_n onto $(k' \cap B) \cup a_n a$.
6. Collapse B from ∂B to the point a_n.
7. Collapse $a_n a$ to a.

Thus $M \searrow^s 0$.

To complete the proof of Theorem A, we see that since N is a regular neighborhood of a collapsible 2-complex, N is a 3-cell. Then L is also a 3-cell, since it is obtained from N by removing two disjoint closed 3-simplexes each with a 2-face in ∂N. Finally, L is simplicially collapsible since it collapses to M, which collapses simplicially to 0.

To prove Theorem B we observe that the L of Theorem A can be collapsed simplicially to a vertex $v \in k'$ which lies on the boundary of L. We first take the wedge at v of $n-1$ copies of L. We can then thicken the point of intersection to a ball to obtain L_n. The intersections of the knots in these copies of L are arranged so that in L_n we have the sum of $n-1$ 2-bridge knots, which is an n-bridge knot K_n with $n-1$ interior 1-simplexes. The collapse of L_n is carried out for the copies of L as in the proof of Theorem A.

References

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