NOETHERIANNESS OF RINGS OF HOLOMORPHIC
FUNCTIONS ON STEIN COMPACT SUBSETS

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In [2] Frisch used the following to prove that for a holomorphic
map φ the set of points where a coherent complex-analytic sheaf is
not φ-flat is a complex-analytic subvariety:

Suppose A is a Stein compact subset of a real- or complex-
analytic space \((X, \emptyset)\). If A is semianalytic, then \(\Gamma(A, \emptyset)\) is
Noetherian.

A counterexample was constructed by Frisch in [2] to show that (1)
is in general false if A is not semianalytic. In this note we give a
necessary and sufficient condition for a Stein compact subset A of a
complex-analytic space \((X, \emptyset)\) so that \(\Gamma(A, \emptyset)\) is Noetherian (and
also an analogue for real-analytic spaces):

**Theorem 1.** Suppose A is a Stein compact subset of a (not necessarily
reduced) complex-analytic space \((X, \emptyset)\), i.e. A admits a neighborhood
basis in X consisting of Stein open subsets of X. Then \(\Gamma(A, \emptyset)\) is
Noetherian if and only if \(Y \cap A\) has only a finite number of topological
components for every complex-analytic subvariety Y defined in any open
neighborhood of A.

If \(E\) is a metric space, then \(d_E\) denotes the metric of \(E\). For \(x \in E\)
and \(A, C \subseteq E\), \(d_E(x, A) = \inf_{y \in A} d_E(x, y)\) and \(d_E(C, A) = \inf_{y \in C} d_E(y, A)\).
For \(\epsilon > 0\) and \(x \in E\), 
\(B_E(x; \epsilon) = \{y \in E | d_E(x, y) < \epsilon\}\).
For \(\epsilon > 0\) and 
\(x, y \in E\) a finite sequence of points \(x_0, \ldots, x_m\) in \(E\) is called an \(\epsilon\)-chain
joining \(x\) to \(y\) if \(x_0 = x, x_m = y\) and \(d_E(x_i, x_{i+1}) < \epsilon\) for \(0 \leq i < m\).
By [4, 5, T(b), p. 169], we have:

If \(x\) and \(y\) are points of a compact subset \(K\) of a metric space, then \(x\) and \(y\) are in the same topological component of \(K\) if and only if, for every \(\epsilon > 0\), \(x\) can be joined to \(y\) by an \(\epsilon\)-chain in \(K\).

**Lemma 1.** Suppose \(E\) and \(F\) are metric spaces and \(f: E \to F\) is con-
tinuous, proper, open, and nowhere degenerate (i.e. inverse images of
discrete sets are discrete). If \(K\) is a connected compact subset of \(F\), then
\(f^{-1}(K)\) has only a finite number of topological components.

**Proof.** We can assume w.l.o.g. that \(K = F\). Since \(f\) is proper and

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open, \( f(E) \) is both open and closed in \( F \). \( f(E) = F \). First we prove the following:

\[ (3) \quad \text{For every } \epsilon > 0 \text{ there exists } \delta > 0 \text{ such that, for every } a \in E, f(B_F(a; \epsilon)) \supset B_F(f(a); \delta) \]

If (3) is not true, then there exist (i) \( \epsilon > 0 \), (ii) a strictly decreasing sequence of positive numbers \( \{ \delta_n \} \) approaching 0, and (iii) a sequence \( \{ a_n \} \) in \( E \) such that \( f(B_F(a_n; \epsilon)) \supset B_F(f(a_n); \delta_n) \). Take \( x_n \in B_F(f(a_n); \delta_n) - f(B_F(a_n; \epsilon)) \). Since \( E \) and \( F \) are both compact, \( x_n \to x^* \) in \( F \) and \( a_n \to a^* \) in \( E \) for some subsequence \( \{ n_m \} \) of \( \{ n \} \).

\( f(a^*) = x^* \), because \( d_F(f(a_n), x_n) < \delta_n \) and \( \delta_n \to 0 \). Since \( B_F(a_n; \epsilon/2) \) and \( f(B_F(a_n; \epsilon/2)) \) are open neighborhoods of \( a^* \) and \( x^* \) respectively, there exists \( m_0 \) such that \( a_{n_m} \in B_F(a^*; \epsilon/2) \) and \( x_{n_m} \in f(B_F(a^*; \epsilon/2)) \) for \( m \geq m_0 \). Hence, for \( m \geq m_0, f(B_F(a_{n_m}; \epsilon)) \supset f(B_F(a^*; \epsilon/2)) \supset x_{n_m} \). Contradiction. (3) is proved.

Fix \( y \in F \). \( f^{-1}(y) \) is finite. Let \( f^{-1}(y) = \{ b_1, \ldots, b_k \} \). Take arbitrarily \( a \in E \). We are going to prove the following:

\[ (4) \quad \text{For any natural number } n \text{ there exists } 1 \leq j_n \leq k \text{ such that } a \text{ can be joined by some } 1/n\text{-chain in } E \text{ to } b_{j_n} \]

Fix a natural number \( n \). By (3) there exists \( \delta > 0 \) such that

\[ (5) \quad f(B_F(c; 1/n)) \supset B_F(f(c); \delta) \quad \text{for every } c \in E \]

Since \( F \) is compact and connected, by (2) there exists a \( \delta \)-chain \( y_0, \ldots, y_m \) in \( F \) joining \( f(a) \) to \( y \). By (5) and by induction on \( i \) we can find \( z_i \in F \) for \( 0 \leq i \leq m \) such that \( z_0 = a, f(z_i) = y_i, \) and \( d_E(z_{i-1}, z_i) < 1/n \) for \( 1 \leq i \leq m \). \( f(z_m) = y_m = y \) implies that \( z_m = b_{j_n} \) for some \( 1 \leq j_n \leq k \). (4) is proved. There exists \( 1 \leq j^* \leq k \) such that \( j^* = j_n \) for an infinite number of natural numbers \( n \). Hence, for any \( \epsilon > 0 \), \( a \) can be joined to \( j^* \) by some \( \epsilon \)-chain in \( E \). By (2) \( a \) is in the same topological component of \( E \) as \( b_{j^*} \). \( E \) has only a finite number of topological components. Q.E.D.

**Lemma 2.** Suppose \( E \) is a compact metric space and \( A \) is a nonempty compact topological component of an open subset \( U \) of \( E \). Then \( A \) is a topological component of \( E \).

**Proof.** Let \( A \) be the topological component of \( E \) containing \( A \). We need only prove that \( A \subset U \). Suppose not. Fix \( b \in A - U \) and \( a \in A \). \( \epsilon = d(A, E - U) > 0 \). Let \( C = \{ x \in E | \epsilon/3 \leq d(x, A) \leq 2\epsilon/3 \} \) and \( D = \{ x \in E | d(x, A) \leq 2\epsilon/3 \} \). We are going to prove the following:
If \( n \) is any natural number \( >3/\varepsilon \), then \( a \) can be joined to some \( c_n \) in \( C \) by some \( 1/n \)-chain in \( D \).

Take \( n>3/\varepsilon \). \( a \) can be joined to \( b \) by some \( 1/n \)-chain \( x_0, \ldots, x_m \) in \( E \). Let \( i \) be the smallest integer such that \( x_i \in D \). \( 0<i\leq m \). \( d(x_{i-1}, x_i) <1/n <\varepsilon/3 \) implies that \( x_{i-1} \in C \). Set \( c_n = x_{i-1} \). Then \( x_0, \ldots, x_{i-1} \) is an \( 1/n \)-chain in \( D \) joining \( a \) to \( c_n \in C \). (6) is proved. Since \( C \) is compact, \( c_{n_m} \to c^* \) in \( C \) for some subsequence \( \{ n_m \} \) of \( \{ n \} \). For any \( \delta >0 \), \( a \) can be joined to \( c^* \) by some \( \delta \)-chain in \( D \). By (2) \( c^* \) belongs to the topological component of \( D \) containing \( a \). Since \( D \subseteq U \) and \( A \) is the topological component of \( U \) containing \( a \), \( c^* \in A \), contradicting \( d(c^*, A) \geq \varepsilon/3 \). Q.E.D.

**Lemma 3.** Suppose \( X \) and \( Y \) are connected, reduced, normal complex-analytic spaces of dimension \( n \). If \( f: X \to Y \) is a nowhere degenerate proper holomorphic map. Then \( f \) is open.

**Proof.** Take \( x \in X \). Let \( D \) be an open neighborhood of \( x \) in \( X \). We have to show that \( f(D) \) is a neighborhood of \( f(x) \). We can assume w.l.o.g. that the closure \( D^- \) of \( D \) is compact and \( D^- \cap f^{-1}(f(x)) = \{ x \} \). \( f(x) \notin f(\partial D) \), where \( \partial D \) is the boundary of \( D \). Since \( Y \) is normal, there is an open neighborhood \( G \) of \( f(x) \) such that \( G \cap f(\partial D) = \emptyset \) and \( G \) is an irreducible complex-analytic space. Let \( H = D \cap f^{-1}(G) \). Let \( g: H \to G \) be induced by \( f \). Then \( g \) is proper and nowhere degenerate. By the proper mapping theorem \([3, \text{V.C.5, p. 162}]\), \( f(H) \) is an \( n \)-dimensional complex-analytic subvariety of \( G \). Since \( G \) is an \( n \)-dimensional irreducible complex-analytic space, \( f(H) = G \). Q.E.D.

**Lemma 4.** Suppose \( A \) is a compact subset of a complex-analytic space \( X \) such that \( Y \cap A \) has only a finite number of topological components if \( Y \) is a complex analytic subvariety in some open neighborhood of \( A \) in \( X \). Suppose \( U \) is an open neighborhood of \( A \) in \( X \) and \( f: Z \to U \) is a proper nowhere degenerate holomorphic map, where \( Z \) is a complex-analytic space. Then \( f^{-1}(A) \) has only a finite number of topological components.

**Proof.** We can assume w.l.o.g. that \( X \) and \( Z \) are reduced and by virtue of the compactness of \( A \) also that \( \dim Z < \infty \). We use induction on \( \dim Z \). The case \( \dim Z = 0 \) is trivial. Suppose \( n \geq 1 \) and the lemma is true for \( \dim Z < n \). Suppose \( \dim Z = n \). Let \( \pi: \hat{Z} \to Z \) be the normalization of \( Z \). We need only prove that \( (f \circ \pi)^{-1}(A) \) has only a finite number of topological components. Hence we can assume w.l.o.g. that \( Z \) is normal. Since \( f^{-1}(A) \) intersects only a finite number of branches of \( Z \), we can assume that \( Z \) is connected. \( V = f(Z) \) is an irreducible complex analytic subvariety of dimension \( n \) in \( U \). Let
σ: \( \tilde{V} \to V \) be the normalization of \( V \). There is a unique nowhere degenerate proper holomorphic map \( \tilde{f}: \tilde{Z} \to \tilde{V} \) such that \( \sigma \circ \tilde{f} = f \). By Lemma 3 \( \tilde{f} \) is open. By Lemma 1 we need only prove that \( \sigma^{-1}(A) \) has only a finite number of topological components. Let \( \tilde{W} \) be the set of all singular points of \( V \). Let \( \tilde{W} = \sigma^{-1}(W) \) and \( \tau = \sigma|_{\tilde{W}} \). Since \( \text{dim } \tilde{W} < \infty \), by induction hypothesis \( \tau^{-1}(A) \) has only a finite number of topological components \( D_1, \ldots, D_k \). Let \( \sigma^{-1}(A) = \bigcup_{i \in I} E_i \) be the decomposition into topological components. Since \( D_i \) is a connected subset of \( \sigma^{-1}(A) \), for some \( i_j \in I, D_j \subset E_{i_j} \). Let \( J = I - \{i_1, \ldots, i_k\} \). Then \( E_i \cap \tilde{W} = \emptyset \) for \( i \in J \). For \( i \in J, E_i \) is a topological component of \( \sigma^{-1}(A) - \tilde{W} \). Since \( \tilde{V} - \tilde{W} \) is biholomorphic to \( V - W, \sigma(E_i), i \in J, \) are distinct topological components of \( A \cap V - W \). By Lemma 2, \( \sigma(E_i), i \in J, \) are distinct topological components of \( A \cap V \). Hence \( J \) is finite. \( I \) is finite. Q.E.D.

Proof of Theorem 1. (a) The "only if" part. Suppose \( V \) is a complex-analytic subvariety of an open neighborhood \( U \) of \( A \) in \( X \) such that \( V \cap A \) has an infinite number of topological components. We are going to construct by induction on \( n \) subsets \( V_n \) of \( V \cap A \) such that:

(i) \( V_1 = V \cap A \),

(ii) \( V_{n+1} \) is a proper open and closed subset of \( V_n \), and

(iii) \( V_n \) has an infinite number of topological components.

Set \( V_1 = V \cap A \). Suppose we have constructed \( V_n \). Since \( V_n \) is not connected, \( V_n \) is the disjoint union of two proper open and closed subsets \( B \) and \( C \) of \( V_n \). Since \( V_n \) has an infinite number of topological components, one of \( B \) and \( C \), say \( B \), has an infinite number of topological components. Set \( V_{n+1} = B \). The construction is complete. Let \( I_n = \{ f \in \Gamma(A, \mathcal{O}) \mid f_x \text{ is not a unit of } \mathcal{O}_x \text{ for } x \in V_n \} \). \( I_n \) is an ideal in the ring \( \Gamma(A, \mathcal{O}) \). \( I_n \subset I_{n+1} \). To prove that \( \Gamma(A, \mathcal{O}) \) is not Noetherian, we need only show

\[
(7) \quad I_n \not\subset I_{n+1}.
\]

Fix \( n \). \( V_{n+1} \) and \( V \cap A - V_{n+1} \) are two disjoint compact subsets of \( U \). Let \( G \) and \( H \) be two disjoint open subsets of \( U \) containing respectively \( V_{n+1} \) and \( V \cap A - V_{n+1} \). \( (U - V) \cup G \cup H \) is an open neighborhood of \( A \). There exists a Stein open neighborhood \( D \) of \( A \) in \( (U - V) \cup G \cup H \). \( D \cap V \subset G \cup H \). Let \( \mathcal{O} \) be the ideal-sheaf of \( V \) on \( U \). Let \( s \in \Gamma(D \cap V, \mathcal{O}/\mathcal{O}) \) be induced by the element \( \bar{s} \in \Gamma(G \cup H, \mathcal{O}) \) which is identically 0 on \( G \) and is identically 1 on \( H \). Since \( D \) is Stein, by Cartan's Theorem B (for not necessarily reduced complex-analytic Stein spaces) \( s \) is induced by some \( \bar{t} \in \Gamma(D, \mathcal{O}) \). Let \( t = \bar{t} \mid A \). Take \( x \in V_n - V_{n+1} \). \( t_x \) is not a unit of \( \mathcal{O}_x \) for \( y \in V_{n+1} \), but \( t_x \) is a unit of \( \mathcal{O}_x \). \( t \in I_{n+1} - I_n \). (7) is proved.
(b) The “if” part. We can assume w.l.o.g. that \( n = \dim X < \infty \). Suppose that \( I \) is an ideal in the ring \( \Gamma(A, \emptyset) \). For \( 0 \leq k \leq n+1 \) consider

\[
\text{For some open neighborhood } U_k \text{ of } A \text{ in } X \text{ there exist } f_1^{(k)}, \ldots, f_r^{(k)} \in \Gamma(U_k, \emptyset) \text{ such that (i) } (f_i^{(k)}|A) \subseteq I, \text{ (ii) if } f \in \Gamma(U, \emptyset) \text{ for } (8)_k \text{ some open neighborhood } U \text{ of } A \text{ in } U_k \text{ and } (f|A) \subseteq I, \text{ then for some open neighborhood } U' \text{ of } A \text{ in } U \text{ the complex-analytic subvariety } \{ x \in U' | f_x \in \sum_{i=1}^{r} \mathcal{O}_x(f_i^{(k)}) \} \text{ is of dimension } < k.
\]

We are going to prove \((8)_k\) by backward induction on \( k \). \((8)_{n+1}\) is obviously true. Suppose \((8)_k\) holds for \( m < k \leq n+1 \). On \( U_{m+1} \) let \( \mathcal{I} = \sum_{i=1}^{r} f_i^{(m+1)} \mathcal{O}_m^{(m+1)} \). Let \( \mathcal{G} \) be the ideal-sheaf on \( U_{m+1} \) defined as follows: for \( x \in U_{m+1} \), \( \mathcal{G}_x = \{ s \in \mathcal{O}_x \} \) for some open neighborhood \( D \) of \( x \) in \( U_{m+1} \) there exist \( t \in \Gamma(D, \emptyset) \) and a complex-analytic subvariety \( V \) of dimension \( \leq m \) in \( D \) such that \( t_y \in \mathcal{G}_y \) for \( y \in U_{m+1} - V \) and \( t_x = s \). \( \mathcal{G} \) is coherent and \( Y = \{ x \in U_{m+1} | \mathcal{G}_x \neq 0 \} \) is a complex-analytic subvariety of dimension \( \leq m \) in \( U_{m+1} \) ([6, Theorem 3]; cf. [7, Satz 3]).

Give \( Y \) the reduced complex-analytic structure. Let \( \pi : Y \to Y \) be the normalization of \( Y \). By Lemma 4, \( \pi^{-1}(A \cap Y) \) has only a finite number of topological components \( C_1, \ldots, C_p \). Take \( z_j \in C_j \) and let \( x_j = \pi(z_j), 1 \leq j \leq p \). Since, for \( x \in A, \mathcal{O}_x \) is Noetherian, for some open neighborhood \( U_m \) of \( A \) in \( U_{m+1} \) there exist \( g_1, \ldots, g_r \in \Gamma(U_m, \emptyset) \) such that \( (g_i|A) \subseteq I \) for \( 1 \leq i \leq r \) and

\[
\{(g_i)|z_j| 1 \leq i \leq r\} \text{ generates the same ideal in } \mathcal{O}_{z_j} \text{ as } \{(f_i)|f \in I\}, \quad 1 \leq j \leq p.
\]

Set \( q(m) = q(m+1) + r \). Define \( f_i^{(m+1)} = f_i^{(m+1)}|U_m \) for \( 1 \leq i \leq q(m+1) \) and \( f_j^{(m+1)} = f_j \) for \( 1 \leq j \leq r \). We claim that these satisfy \((8)_m\). Take \( f \in \Gamma(U, \emptyset) \) for some open neighborhood \( U \) of \( A \) in \( U_{m+1} \) such that \( (f|A) \subseteq I \). By \((8)_{m+1}\), for some open neighborhood \( U'' \) of \( A \) in \( U \), \( (f|U'') \in \Gamma(U'', \mathcal{G}) \). The complex-analytic subvariety

\[
Z = \{ x \in U'' | f_x \in \sum_{i=1}^{q(m)} \mathcal{O}_x(f_i^{(m)}) \}
\]

is contained in \( Y \). Let \( Z_i \) be the union of \( m \)-dimensional branches of \( Z \) not intersecting \( A \). Let \( U'' = U'' - Z_1 \) and \( Z_0 = Z \cap U'' \). All we have to prove is that \( Z_0 \) has no \( m \)-dimensional branch. Suppose the contrary. Let \( V \) be an \( m \)-dimensional branch of \( Z_0 \). \( \pi^{-1}(V) \cap C_j \neq \emptyset \) for some \( 1 \leq j \leq p \). Let \( \pi^{-1}(U'' \cap Y) = \bigcup_{\lambda \in L} Y_\lambda \) be the decomposition into irreducible branches. Since \( \dim V = m \) and \( \dim Y \leq m \), \( \pi^{-1}(V) = \bigcup_{\lambda \in M} Y_\lambda \) for some subset \( M \) of \( L \). \( Y_\lambda \cap C_j \neq \emptyset \) for some \( \lambda \in M \). Since
$Y_x \cap \pi^{-1}(A \cap Y)$ is a union of topological components of $\pi^{-1}(A \cap Y)$, $C_i \subset Y_x \cap \pi^{-1}(A \cap Y)$. $z_i \in \pi^{-1}(V)$. $x_i \in V$, contradicting (9). Hence (8) is true for $0 \leq k \leq n+1$.

We claim that $\{\langle i_0(0) \rangle \mid 1 \leq i \leq q(0)\}$ generates $I$ over $\Gamma(A, \emptyset)$. Take $g \in I$. Then $g=\hat{f}(A)$ for some $f \in \Gamma(U, \emptyset)$, where $U$ is an open neighborhood of $A$ in $U$. By (8) there exists an open neighborhood $U'$ of $A$ in $U$. Let $W$ be a Stein open neighborhood of $A$ in $U'$. Let $\phi: \emptyset(\emptyset) \to \sum_{i=1}^{q(0)} \emptyset i_i(0)$ be the sheaf-epimorphism on $W$ defined by $\phi(s_1, \ldots, s_q(0)) = \sum_{i=1}^{q(0)} s_i(0)i$ for $(s_1, \ldots, s_q(0)) \in \emptyset(0)$ and $x \in W$. Since $H^1(W, \text{Ker } \phi) = 0$, there exist $a_1, \ldots, a_q(0) \in \Gamma(W, \emptyset)$ such that $f = \sum_{i=1}^{q(0)} a_q(0)i_i$ on $W$. Let $b_i = a_i|A$, $1 \leq i \leq q(0)$. Then $g = \sum_{i=1}^{q(0)} b_i(i_i(0))$. Q.E.D.

**Corollary.** Suppose $A$ is a Stein compact subset of a complex-analytic space $(X, \emptyset)$ and $A$ is contained in an 1-dimensional complex-analytic subvariety of $X$. Then $\Gamma(A, \emptyset)$ is Noetherian if and only if $A$ has only a finite number of topological components.

We are going to give a real-analytic analogue of Theorem 1.

**Lemma 5.** Suppose $A$ is a compact subset of $\mathbb{R}^n$. Then $A$ is a Stein compact subset of $\mathbb{C}^n$.

**Proof.** Take an open neighborhood $U$ of $A$ in $\mathbb{C}^n$. We have to prove that $A$ has a Stein open neighborhood in $U$. Let $W = U \cap \mathbb{R}^n$. For some real-analytic functions $g_1, \ldots, g_k$ on $W$ the map $(g_1, \ldots, g_k): W \to \mathbb{R}^k$ imbeds $W$ as a real-analytic submanifold of $\mathbb{R}^k$ [1, Folgerung zu Satz 8, p. 53].

For some open neighborhood $G$ of $W$ in $U$, $g_1, \ldots, g_k$ are the restriction to $W$ of holomorphic functions $f_1, \ldots, f_k$ on $G$. Take $c > \sup \{|g_j(x)| \mid x \in A, 1 \leq j \leq k\}$. Let $K = \{x \in W \mid |g_j(x)| \leq c, 1 \leq j \leq k\}$. Let $H$ be a relatively compact open neighborhood of $K$ in $G$. Let $z_1 = x_1 + i y_1, \ldots, z_n = x_n + i y_n$ be the coordinates of $\mathbb{C}^n$. For $\epsilon > 0$ let

$$D_\epsilon = \{x \in G \mid |y_1(x)| < \epsilon, \ldots, |y_n(x)| < \epsilon, |f_1(x)| < \epsilon, \ldots, |f_k(x)| < \epsilon\}.
$$

$D_\epsilon$ is relatively compact in $G$ for some $\delta > 0$, because $\cap_{\epsilon > 0}(D_\delta \cap H) = K \subset H$ [4, 5, F(a), p. 163]. $D_\epsilon$ is a Stein open neighborhood of $A$ in $U$. Q.E.D.

**Theorem 2.** Suppose $A$ is a compact subset of a coherent real-analytic space $(X, \emptyset)$ [1, p. 44]. Then $\Gamma(A, \emptyset)$ is Noetherian if and only if $Z \cap A$ has only a finite number of topological components for every real-analytic subvariety $Z$ of an open neighborhood $U$ of $A$ in $X$ definable by a coherent ideal-sheaf on $U$. 
Proof. (a) The “if” part. W.l.o.g. we can suppose that $X$ is a coherent real-analytic subvariety of $\mathbb{R}^n$ for some $n$ [1, Folgerung zu Satz 8, p. 53]. Let $\mathfrak{s}$ be the ideal-sheaf of $X$ on $\mathbb{R}^n$. Let $\mathfrak{A}$ and $\mathfrak{O}$ be respectively the structure-sheaves of $\mathbb{R}^n$ and $\mathbb{C}^n$. By Lemma 5 $A$ is a Stein compact subset of $\mathbb{C}^n$. Let $Y$ be an arbitrary complex-analytic subvariety in some open neighborhood $U$ of $A$ in $\mathbb{C}^n$. Let $G$ be a Stein open neighborhood of $A$ in $U$. Let $G$ be a relatively compact open neighborhood of $A$ in $\mathcal{G}$. $Y \cap G$ can be defined by a finite number of holomorphic functions $f_1, \ldots, f_k$ on $G$. Let $W = G \cap \mathbb{R}^n$ and $Z = Y \cap W$. $Z$ is a real-analytic subvariety of $W$ definable by a coherent ideal-sheaf on $W$, because $Z$ is the set of common zeros of the real parts and imaginary parts of the restrictions of $f_1, \ldots, f_k$ to $W$. $Y \cap A = Z \cap A$ has only a finite number of topological components. By Theorem 1 $\Gamma(A, 0)$ is Noetherian. Since on $A \mathfrak{A} = \mathfrak{A} \otimes_\mathbb{R} \mathfrak{O}$, $\Gamma(A, \mathfrak{A})$ is Noetherian. Since for any open neighborhood $D$ of $A$ in $\mathbb{R}^n$, $H^1(D, \mathfrak{A}) = 0$ [1, Satz 9, p. 54], $\Gamma(A, \mathfrak{A}) \rightarrow \Gamma(A, \mathfrak{A})$ is surjective. Hence $\Gamma(A, 0\mathfrak{A})$ is Noetherian.

(b) The “only if” part is proved in exactly the same way as that of Theorem 1 except that instead of Cartan’s Theorem B, Satz 9 of [1] is used. Q.E.D.

Corollary. Suppose $A$ is a compact subset of a coherent real-analytic space $\Gamma(X, \mathfrak{A})$ and $A$ is contained in an 1-dimensional real-analytic subvariety of $X$ definable by a coherent ideal-sheaf. Then $\Gamma(A, \mathfrak{A})$ is Noetherian if and only if $A$ has only a finite number of topological components.

Remark. (1) follows from Theorems 1 and 2 and the fact that a compact semianalytic set has only a finite number of topological components (see [5]).

References


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