CORRECTION TO "AN APPLICATION OF GRAPH THEORY TO ALGEBRA"

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Mr. L. H. Harper recently pointed out to me that the proof in [1] is not complete. The argument given in Cases 2 and 3 is only valid if $P \neq A$ or B. The purpose of this note is to supply the missing details. We use the notation of [1] and assume all the hypotheses of [1, §5].

LEMMA. *Suppose* Γ *has a vertex* X *of order* 2 *such that the two edges* e *and* e' *meeting* X *join* P *to* X *and* X *to* P *respectively. Then the theorem is true for* Γ.

PROOF. Let Γ' be the result of deleting e, e', and X. The theorem holds for Γ' by induction. Any unicursal path on Γ' has the form $\pi_1\pi_2 \cdots \pi_n$ where each π_i is a path starting and ending at P but not meeting P between. Clearly n is the number of edges leaving P in Γ' and so is the same for all paths. Let λ be the path ee' from P to P in Γ. We get all possible unicursal paths on Γ by starting with such paths on Γ' and inserting λ, getting $\lambda\pi_1 \cdots \pi_n$, $\pi_1\lambda \cdots \pi_n, \cdots$, $\pi_1 \cdots \pi_n\lambda$. Assuming that e, e' are the last two edges in the chosen ordering of the edges, we have $\epsilon(\pi_1 \cdots \pi_i\lambda\pi_{i+1} \cdots \pi_n) = \epsilon(\pi_1 \cdots \pi_n)$. Thus $\sum\epsilon(\pi) = (n+1)\sum\epsilon(\pi') = 0$, the first sum being over all unicursal paths on Γ and the second over such paths on Γ'.

We now consider Case 2 of [1]. If $P = A$, we can repeat the argument of Case 2 using B and C in place of B and A with only minor modifications. This will be possible provided $C \neq A$, but if $C = A$, the lemma applies with $X = B$. Suppose now that $P = B$. Let U be the set of unicursal paths on Γ starting at A, U' the set of such paths which begin with e, and U_i the set of unicursal paths on Γ_i starting at A. Then the argument of [1, Case 2] shows that $U = U' \cup \bigcup U_i$, a disjoint union. Since the theorem holds for U by what we have just proved, and also for U_i, we see that $\sum\epsilon(\pi') = 0$ where π' runs over all elements of U'. But there is a one-to-one correspondence between U' and the set of unicursal paths starting from B, given by $ee'e_1 \cdots e_n \leftrightarrow e'e_1 \cdots e_n e$. Since $n = E - 2$ is even, $\epsilon(ee'e_1 \cdots e_n) = -\epsilon(e' \cdots e_n e)$. Therefore the theorem holds in this case also.

Finally, we consider Case 3. If there is an edge not meeting P, choose it for e_4. Then $P \neq A$, B and we are done. Suppose every edge meets P. Let P, $A_1, \cdots, A_n$ be the vertices. Then $E = 2V = 2n+2$.

Received by the editors July 30, 1968.

Each A_i must be joined to P by at least two edges. This uses up all but two edges which must either be loops from P to P, or must both join P to some A_i. In this case, the lemma clearly applies except in the trivial cases $V=1$ or 2.

There is also a misprint in Figure 9 of [1]. This figure should contain an additional edge labelled e_3 with A as initial point.

Reference

1. R. G. Swan, *An application of graph theory to algebra*, Proc. Amer. Math. Soc. **14** (1963), 367–373.

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