A NOTE ON A THEOREM OF JACOBSON

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The question as to whether every derivation of a simple algebra is inner, is still unsettled. The simple proofs given below of Theorem A, which is a special case of a well-known theorem (see [4, pp. 22–23]), and that of Theorem B would possibly be a new approach to this question.

Theorem A. If $A$ is a simple algebra with identity over an algebraically closed field $F$ of characteristic zero, then every derivation of $A$ is inner.

Proof. In the first instance, let $A$ have an identity or not. Let $R_x$ ($L_x$) denote the right (left) multiplication in $A$, and $D$ be a derivation of $A$. Then $A$ has neither proper ideals nor proper $D$-ideals. In other words, the Lie algebra $L$ ($L'$) generated by $R_x$, $L_x$ ($R_x$, $L_x$ and the derivation $D$) is irreducible. Moreover, $L' = L + \{\alpha D\}_{\alpha \in F}$ (a vector space sum and not necessarily a direct sum), and $L$ is an ideal of $L'$. Further, by a theorem of Jacobson [1, p. 47], we have $L = C \oplus [L, L]$; $L' = C' \oplus [L', L']$ for centers $C, C'$ of $L, L'$; $[L, L], [L', L']$ are semisimple; $[L, L]$ is an ideal of $[L', L']$. Any transformation $T$ in $C$ commutes with the irreducible associative algebra generated by $R_x$, $L_x$ and hence should be a multiple of the identity transformation $I$, by Schur's lemma. Now, if $A$ contains an identity, $C = F_1$, $C = F_1$; since the dimension of $L'$ is at the most greater by unity than that of $L$, and since $[L', L']$ cannot have a one dimensional (abelian) ideal complementary to $[L, L]$, $[L, L] = [L', L']$; i.e., $L = L'$, or, $D \subseteq L$. Thus, every derivation of $A$ is inner.

Remark 1. In case $A$ is any simple nonassociative algebra, then $C \subseteq C'$ and therefore $[L, L] = [L', L']$ in this case also. If, in addition $C = C'$ for every derivation $D$ of $A$, then every derivation of $A$ will be inner. Because $F$ is algebraically closed, $C = 0$ or $C = F_1$, $C' = 0$ or $C' = F_1$. Since $C = F_1$ implies $C' = F_1 = C$ and since every derivation is inner in this case as well as when $C' = 0$, the question raised at the outset boils down to the consideration of the only case $C = 0$, $C' = F_1$. The plausibility of this case remains to be seen.

Now, in the case of simple Lie algebra $A$ over a field $F$ of characteristic zero, $L = \{\text{ad } x\}_{x \in A}$; $L' = L + \{\alpha D\}_{\alpha \in F}$. Since $A$ is simple, the center of $A = \{x \in A \mid \text{ad } x = 0\} = \{0\}$. If $ad y \in \text{center } C$ of $L$, then

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\[ [\text{ad } x, \text{ad } y] = \text{ad}[x, y] = 0 \] for every \( x \) in \( A \); hence \([x, y] = 0\), i.e., \( \text{ad } y = 0 \) and \( C = 0 \). If \( \text{ad } z + \alpha D \) belongs to the center \( C' \) of \( L' \), then \([\text{ad } z + \alpha D, \text{ad } x] = 0\) for all \( x \), i.e., \( \text{ad}[z, x] = \alpha \text{ ad } x D = 0 \) for all \( x \) in \( A \), or, \([z, x] - \alpha x D = 0\), i.e., \( -(\text{ad } z + \alpha D) = 0 \). Hence \( C' = 0 \). Thus \( L, L' \) are semisimple \([1, p. 47]\). Further arguments as in the proof of Theorem A yield

**Theorem B.** Every derivation of a simple Lie algebra \( A \) over a field of characteristic zero is inner.

**Remark 2.** Theorem B is, more generally, true for a simple Malcev algebra \( A \) \([3]\). For, the center \( C \) of \( L \) is then known to be the zero ideal \([2, \text{Corollary 5.32}]\). A similar argument shows that the center \( C' \) of \( L' \) is also the zero ideal.

**References**


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