A NOTE ON A THEOREM OF JACOBSON

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The question as to whether every derivation of a simple algebra is inner, is still unsettled. The simple proofs given below of Theorem A, which is a special case of a well-known theorem (see [4, pp. 22-23]), and that of Theorem B would possibly be a new approach to this question.

**Theorem A.** If \( A \) is a simple algebra with identity over an algebraically closed field \( F \) of characteristic zero, then every derivation of \( A \) is inner.

**Proof.** In the first instance, let \( A \) have an identity or not. Let \( R_x (L_x) \) denote the right (left) multiplication in \( A \), and \( D \) be a derivation of \( A \). Then \( A \) has neither proper ideals nor proper \( D \)-ideals. In other words, the Lie algebra \( L (L') \) generated by \( R_x, L_x (R_x, L_x \) and the derivation \( D \)) is irreducible. Moreover, \( L' = L + \{ \alpha D \}_{\alpha \in F} \) (a vector space sum and not necessarily a direct sum), and \( L \) is an ideal of \( L' \). Further, by a theorem of Jacobson [1, p. 47], we have \( L = C \oplus [LL]; L' = C' \oplus [LL'] \) for centers \( C, C' \) of \( L, L' \); \( [LL], [LL'] \) are semisimple; \( [LL] \) is an ideal of \( [LL'] \). Any transformation \( T \) in \( C \) commutes with the irreducible associative algebra generated by \( R_x, L_x \) and hence should be a multiple of the identity transformation \( I \), by Schur's lemma. Now, if \( A \) contains an identity, \( C = F \), \( C' = FI \); since the dimension of \( L' \) is at the most greater by unity than that of \( L \), and since \( [LL'] \) cannot have a one dimensional (abelian) ideal complementary to \( [LL] \), \( [LL'] \); i.e., \( L = L' \), or, \( D \subseteq L \). Thus, every derivation of \( A \) is inner.

**Remark 1.** In case \( A \) is any simple nonassociative algebra, then \( C \subseteq C' \) and therefore \( [LL] = [LL'] \) in this case also. If, in addition \( C = C' \) for every derivation \( D \) of \( A \), then every derivation of \( A \) will be inner. Because \( F \) is algebraically closed, \( C = 0 \) or \( C = FI \), \( C' = 0 \) or \( C' = FI \). Since \( C = FI \) implies \( C' = FI = C \) and since every derivation is inner in this case as well as when \( C' = 0 \), the question raised at the outset boils down to the consideration of the only case \( C = 0 \), \( C' = FI \). The plausibility of this case remains to be seen.

Now, in the case of simple Lie algebra \( A \) over a field \( F \) of characteristic zero, \( L = \{ \text{ad } x \}_{x \in A}; L' = L + \{ \alpha D \}_{\alpha \in F} \). Since \( A \) is simple, the center of \( A \) is \( \{ x \in A \mid \text{ad } x = 0 \} = \{ 0 \} \). If \( y \in \text{center } C \) of \( L \), then

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\[\text{ad } x, \text{ ad } y\] = \text{ad}[x, y] = 0 \text{ for every } x \text{ in } A; \text{ hence } [x, y] = 0, \text{ i.e.,}\ad y = 0 \text{ and } C = 0. \text{ If ad } z + \alpha D \text{ belongs to the center } C' \text{ of } L', \text{ then } [\text{ad } z + \alpha D, \text{ ad } x] = 0 \text{ for all } x, \text{ i.e., } \text{ad}[z, x] - \alpha \text{ ad } xD = 0 \text{ for all } x \text{ in } A, \text{ or, } [z, x] - \alpha xD = 0, \text{ i.e., } -(\text{ad } z + \alpha D) = 0. \text{ Hence } C' = 0. \text{ Thus } L, L' \text{ are semisimple [1, p. 47]. Further arguments as in the proof of Theorem A yield}

**Theorem B.** Every derivation of a simple Lie algebra \( A \) over a field of characteristic zero is inner.

**Remark 2.** Theorem B is, more generally, true for a simple Malcev algebra \( A \) [3]. For, the center \( C \) of \( L \) is then known to be the zero ideal [2, Corollary 5.32]. A similar argument shows that the center \( C' \) of \( L' \) is also the zero ideal.

**References**


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