FUNCTIONS THAT OPERATE IN THE FOURIER ALGEBRA OF A COMPACT GROUP

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0. Only real-analytic functions operate in the Fourier algebra of any compact group that has an infinite abelian subgroup. This extends the theorems of Helson, Kahane, Katznelson, and Rudin [4] which apply to the algebra of absolutely convergent Fourier series on compact abelian groups. The Fourier algebra of a locally compact group has been studied by H. Mirkil [6], W. F. Stinespring [9], R. A. Mayer [5], C. Herz [3], and most thoroughly by P. Eymard [1]. We will state here the relevant definitions and facts, and prove that the restriction of the Fourier algebra to a closed subgroup is the Fourier algebra of the subgroup, and use this to lift up the theorem on operating functions.

1. Definitions and notation. Let \( G \) be a compact group, \( \hat{G} \) the set of equivalence classes of irreducible unitary continuous representations (henceforth “representation” means “unitary continuous representation”) and if \( \alpha \in \hat{G} \), then \( T_\alpha \) is some element of the class \( \alpha \). \( T_\alpha \) is an irreducible representation of \( G \) acting on a \( d_\alpha \)-dimensional complex space \( V_\alpha \), and \( \chi_\alpha \), the character of \( \alpha \), is given by \( \chi_\alpha(x) = \text{Tr}(T_\alpha(x)) \), for all \( \alpha \in \hat{G} \) (\( \text{Tr} \equiv \text{Trace} \)). (Naimark’s book [7] is a reference for these statements on \( \hat{G} \) and the Fourier transform.) \( C(G), L^r(G), M(G) \) denote the space of complex continuous functions on \( G \), \( L^r \) with respect to the normalized Haar measure (\( \int_G dx = 1 \)) on \( G \), and the space of regular Borel measures on \( G \) respectively.

For \( f \in L^1(G), \mu \in M(G), \alpha \in \hat{G} \) define
\[
\hat{f}_\alpha = \int_G f(x) T_\alpha(x^{-1}) dx, \quad \hat{\mu}_\alpha = \int_G T_\alpha(x^{-1}) d\mu(x);
\]
these are linear operators on \( V_\alpha \), with operator norms denoted by \( \| f_\alpha \|_\infty \), \( \| \hat{\mu}_\alpha \|_\infty \) respectively, and \( \| f_\alpha \|_\infty \leq \| f \|_1 \), \( \| \hat{\mu}_\alpha \|_\infty \leq \| \mu \| \). If \( f \in L^2(G) \), then \( f \) has the Fourier series \( f(x) \sim \sum_{\alpha \in \hat{G}} d_\alpha \text{Tr}(f \delta_\alpha(x)) \) (\( L^2 \)-convergence) and \( \| f \|_2^2 = \sum_{\alpha \in \hat{G}} d_\alpha \text{Tr}(f^* f_\alpha) \) (\( * \equiv \text{adjoint} \)).

For any linear operator \( U \) on \( V_\alpha \), \( 1 \leq p < \infty \) define \( \| U \|_p = (\text{Tr}(U^* U)^{p/2})^{1/p} = (\text{Tr} |U|^p)^{1/p} \) (see e.g. R. Mayer [5]). Then the Fourier algebra \( A(G) \) may be defined as follows:

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\[ A(G) = \left\{ f \in L^1(G) : \|f\|_A = \sum_{a \in G} d_a |\hat{f}_a|_1 < \infty \right\} . \]

If \( f \in A(G) \) then the inequality
\[ | \text{Tr}(\hat{f}_a T_a(x)) | \leq \|\hat{f}_a\|_1 \| T_a(x) \|_\infty = \|\hat{f}_a\|_1, \]
shows that the Fourier series \( \sum_{a \in G} d_a \text{Tr}(\hat{f}_a T_a(x)) \) converges uniformly and absolutely, thus \( A(G) \subset C(G) \) and \( \|f\|_\infty \leq \|f\|_A \). Note \( \|\hat{f}\|_A = \|\hat{f}\|_A \), so \( A(G) \) is symmetric.

Observe if \( G \) is abelian then \( A(G) = (L^1(\hat{G}))^\wedge \), the space of absolutely convergent Fourier series. P. Eymard has shown that \( A(G) \) is a commutative Banach algebra, with identity, under pointwise multiplication, and to each maximal ideal \( M \) there corresponds \( x_M \in G \) such that \( M = \{ f : f(x_M) = 0 \} \); that is, \( A(G) \) is a symmetric algebra of continuous functions on its maximal ideal space \( G \). This of course shows that if \( \psi \) is real-analytic on a region \( E, f \in A(G) \), and range \( f \subset E \), then \( \psi \circ f \in A(G) \).

We state two lemmas to tie our definition of \( A(G) \) together with Eymard’s. For \( \mu \in M(G) \), let \( \|\mu\|_\infty = \sup_{a \in G} \|\hat{\mu}_a\|_\infty (\leq \|\mu\|) \).

**Lemma 1.** For \( f \in C(G) \)
\[ \|f\|_A = \sup \left\{ \left| \int_G f \, d\mu \right| : \mu \in M(G), \|\mu\|_\infty \leq 1 \right\}. \]

**Lemma 2.**
\[ \|\hat{\mu}\|_\infty = \sup \{ \|f \ast \mu\|_2 : f \in L^2(G), \|f\|_2 \leq 1 \} \]

\((\ast = \text{convolution})\).

2. Induced representations. (For reference see R. Godement [2].) Let \( H \) be a closed subgroup of \( G \) (then \( \hat{H} \) is defined similarly to \( \hat{G} \), and \( \tau \) is an element of the class \( i \in \hat{H} \), with character \( \phi_i \) and dimension \( d_i \)). For \( \alpha \in \hat{G} \), \( T_a \) restricted to \( H \) is a representation of \( H \) hence it is a direct sum of irreducible representations of \( H \), say \( T_a(h) = \sum \Theta_{i \in \hat{H}} n_a(i) \tau_i(h) \) for all \( h \in H \) where \( n_a(i) \) is the number of times that \( T_a \mid_H \) contains \( \tau_i \); for a fixed \( \alpha \), \( n_a(i) > 0 \) for only a finite number of \( i \in \hat{H} \) since \( d_a = \sum_{i \in \hat{H}} n_a(i) d_i \).

For \( i \in \hat{H} \), let \( \phi'_i \) be the element of \( M(G) \) defined by \( \int_H f(h) \phi'_i(h) \, dh \) for all \( f \in C(G) \) (\( dh = \text{normalized Haar measure on } H \)). \( \phi'_i \) is called the generalized character of the representation of \( G \) induced by \( \tau_i \). Now for \( \alpha \in \hat{G} \), \( (\phi'_i)_{\alpha} = \int_H T_a(h^{-1}) \phi_i(h) \, dh \), thus \( (\phi'_i)_{\alpha} \neq 0 \) if and only if \( n_a(i) > 0 \), but \( \phi'_i \neq 0 \) in \( M(G) \); therefore there exists at least one \( \alpha \in \hat{G} \) such that \( n_a(i) > 0 \). Let \( \rho \) be the restriction map: \( C(G) \rightarrow C(H) \), then \( \rho^* \) is an injection: \( M(H) \rightarrow M(G) \).
Lemma 3. Let $\mu \in \mathcal{M}(H)$, then $\|\hat{\mu}\|_{\infty} = \|\hat{\mu} \circ \mu\|_{\infty}$.

Proof.

$$\|\hat{\mu} \circ \mu\|_{\infty} = \sup_{\alpha \in \hat{G}} \left\| \int_H T_{\alpha}(h^{-1}) d\mu(h) \right\|_{\infty}$$

$$= \sup_{\alpha \in \hat{G}} \left\| \sum_{i \in \hat{H}} \otimes n_{\alpha}(i) \hat{\mu}_i \right\|_{\infty}$$

$$= \sup_{\alpha \in \hat{G}} \max_{n_{\alpha}(i) > 0} \|\hat{\mu}_i\|_{\infty} = \sup_{i \in \hat{H}} \|\hat{\mu}_i\|_{\infty} = \|\hat{\mu}\|_{\infty}. \quad \text{Q.E.D.}$$

3. Main theorems.

Theorem 1. $\mathcal{A}(G) \mid H = \mathcal{A}(H)$; if $F \in \mathcal{A}(G)$ then $\rho F \in \mathcal{A}(H)$ and $\|\rho F\|_{\mathcal{A}} \leq \|F\|_{\mathcal{A}}$, and for each [real] $f \in \mathcal{A}(H)$ there exists [real] $F \in \mathcal{A}(G)$ such that $\rho F = f$ and $\|F\|_{\mathcal{A}} = \|f\|_{\mathcal{A}}$.

Proof. Let $F \in \mathcal{A}(G)$, then

$$\|\rho F\|_{\mathcal{A}} = \sup \left\{ \left\| \int_H F d\mu \right\| : \mu \in \mathcal{M}(H), \|\mu\|_{\infty} \leq 1 \right\}$$

(by Lemma 1)

$$= \sup \left\{ \left\| \int_G F d(\rho^\ast \mu) \right\| : \mu \in \mathcal{M}(H), \|\rho^\ast \mu\|_{\infty} \leq 1 \right\}$$

(by Lemma 3)

$$\leq \sup \left\{ \left\| \int_G F d\lambda \right\| : \lambda \in \mathcal{M}(G), \|\lambda\|_{\infty} \leq 1 \right\} = \|F\|_{\mathcal{A}}.$$
\[ \|F\|_A \leq \sum_{i \in \mathbb{H}} \|F_i\|_1 = \sum_{i \in \mathbb{H}} d_i \|f_i\|_1 = \|f\|_A. \]

But \( \rho F = f \), so \( \|F\|_A \geq \|f\|_A \), thus \( \|F\|_A = \|f\|_A \). Further if \( f \) is real, then so is \( F \), because \( \alpha(\mathbb{H}) = \mathbb{H} \). Q.E.D.

Let \( G \) be the class of compact groups having infinite abelian subgroups, hence compact infinite abelian subgroups (by closure).

**Theorem 2.** If \( G \in G \) then only real-analytic functions operate in \( A(G) \), that is, if \( \psi \) is defined on a closed convex set \( E \) in the complex plane, and \( \psi \circ F \in A(G) \) whenever \( F \in A(G) \) and range \( F \subseteq E \) then \( \psi \) is real-analytic on \( E \).

**Proof.** Let \( H \) be an infinite compact abelian subgroup of \( G \). By Lemma 6.6.2, p. 143 (Rudin [8]), for \( n > 0 \),
\[
\sup \{ \|e^{it}\|_A : \text{real } f \in A(H), \|f\|_A \leq n \} = e^n.
\]

For any \( \epsilon > 0 \), there exists a real \( f \in A(H) \) such that \( \|f\|_A \leq n \), \( \|e^{it}\|_A > e^n - \epsilon \). By Theorem 1, there exists real \( F \in A(G) \) such that \( \rho F = f \) and \( \|F\|_A = \|f\|_A \leq n \). Then \( \rho(e^{it}) = e^{it} \), hence \( e^n \geq \|e^{it}\|_A \geq \|e^{it}\|_A > e^n - \epsilon \), therefore
\[
\sup \{ \|e^{it}\|_A : \text{real } F \in A(G), \|F\|_A \leq n \} = e^n.
\]

Now the proof of Helson, Kahane, Katznelson and Rudin [4] applies to \( A(G) \) to show that only real-analytic functions operate. Q.E.D.

**4. Remarks.** One may want to extend the theory of the Fourier algebra to homogeneous spaces, but there is a slight snag. Here is the situation: Let \( G \) be a compact group, \( H \) a closed subgroup, \( X = G/H \) the homogeneous space of right cosets of \( H \), then let \( A(G/H) = \{ f \in A(G) : f(x) = f(xh), \text{all } x \in G, h \in H \} \). \( A(G/H) \) can be interpreted as a Banach algebra of continuous functions on \( X \) with maximal ideal space \( \mathbb{H} \times X \). Now suppose that another group \( G \), also acts transitively on \( X \) and \( X \cong G_1/H_1 \). Is there a natural isomorphism between \( A(G/H) \) and \( A(G_1/H_1) \)? In general, the answer is no, as may be seen in the case of \( SO(4)/SO(3) \cong Sp(1) \), (the group of unit quaternions). In the case \( G_1 \subseteq G, H_1 = H \cap G_1 \), Theorem 1 shows that \( A(G/H) \subset A(G_1/H_1) \).

**References**


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