FUNCTIONS THAT OPERATE IN THE FOURIER ALGEBRA OF A COMPACT GROUP

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0. Only real-analytic functions operate in the Fourier algebra of any compact group that has an infinite abelian subgroup. This extends the theorems of Helson, Kahane, Katznelson, and Rudin [4] which apply to the algebra of absolutely convergent Fourier series on compact abelian groups. The Fourier algebra of a locally compact group has been studied by H. Mirkil [6], W. F. Stinespring [9], R. A. Mayer [5], C. Herz [3], and most thoroughly by P. Eymard [1]. We will state here the relevant definitions and facts, and prove that the restriction of the Fourier algebra to a closed subgroup is the Fourier algebra of the subgroup, and use this to lift up the theorem on operating functions.

1. **Definitions and notation.** Let G be a compact group, $\hat{G}$ the set of equivalence classes of irreducible unitary continuous representations (henceforth "representation" means "unitary continuous representation") and if $\alpha\in\hat{G}$, then T_α is some element of the class α. T_α is an irreducible representation of G acting on a d_α-dimensional complex space V_α, and χ_α, the character of α, is given by $\chi_\alpha(x)=\mathrm{Tr}(T_\alpha(x))$, for all $\alpha\in\hat{G}$ (Tr$\equiv$Trace). (Naimark's book [7] is a reference for these statements on $\hat{G}$ and the Fourier transform.) $C(G)$, $L^p(G)$, $M(G)$ denote the space of complex continuous functions on G, L^p with respect to the normalized Haar measure ($\int_G dx=1$) on G, and the space of regular Borel measures on G respectively.

For $f\in L^1(G)$, $\mu\in M(G)$, $\alpha\in\hat{G}$ define

$$\hat{f}_\alpha=\int_G f(x)T_\alpha(x^{-1})dx,\qquad \hat{\mu}_\alpha=\int_G T_\alpha(x^{-1})d\mu(x);$$

these are linear operators on V_α, with operator norms denoted by $\|\hat{f}_\alpha\|_\infty$, $\|\hat{\mu}_\alpha\|_\infty$ respectively, and $\|\hat{f}_\alpha\|_\infty\leqq\|f\|_1$, $\|\hat{\mu}_\alpha\|_\infty\leqq\|\mu\|$. If $f\in L^2(G)$, then f has the Fourier series $f(x)\sim\sum_{\alpha\in\hat{G}}d_\alpha\mathrm{Tr}(\hat{f}_\alpha T_\alpha(x))$ (L^2-convergence) and $\|f\|_2^2=\sum_{\alpha\in\hat{G}}d_\alpha\,\mathrm{Tr}(\hat{f}_\alpha^*\hat{f}_\alpha)$ ($*\equiv$adjoint).

For any linear operator U on V_α, $1\leqq p<\infty$ define $\|U\|_p=(\mathrm{Tr}(U^*U)^{p/2})^{1/p}=(\mathrm{Tr}|U|^p)^{1/p}$ (see e.g. R. Mayer [5]). Then the Fourier algebra $A(G)$ may be defined as follows:

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$$A(G) = \left\{ f \in L^1(G) \colon \|f\|_A = \sum_{\alpha \in \hat{G}} d_\alpha \|\hat{f}_\alpha\|_1 < \infty \right\}.$$

If $f \in A(G)$ then the inequality

$$\left| \operatorname{Tr}(\hat{f}_\alpha T_\alpha(x)) \right| \leqq \|\hat{f}_\alpha\|_1 \|T_\alpha(x)\|_\infty = \|\hat{f}_\alpha\|_1$$

shows that the Fourier series $\sum_{\alpha \in \hat{G}} d_\alpha \operatorname{Tr}(\hat{f}_\alpha T_\alpha(x))$ converges uniformly and absolutely, thus $A(G) \subset C(G)$ and $\|f\|_\infty \leqq \|f\|_A$. Note $\|\bar{f}\|_A = \|f\|_A$, so $A(G)$ is symmetric.

Observe if G is abelian then $A(G) = (L^1(\hat{G}))^\wedge$, the space of absolutely convergent Fourier series. P. Eymard has shown that $A(G)$ is a commutative Banach algebra, with identity, under pointwise multiplication, and to each maximal ideal M there corresponds $x_M \in G$ such that $M = \{f \colon f(x_M) = 0\}$; that is, $A(G)$ is a symmetric algebra of continuous functions on its maximal ideal space G. This of course shows that if ψ is real-analytic on a region E, $f \in A(G)$, and range $f \subset E$, then $\psi \circ f \in A(G)$.

We state two lemmas to tie our definition of $A(G)$ together with Eymard's. For $\mu \in M(G)$, let $\|\hat{\mu}\|_\infty = \sup_{\alpha \in \hat{G}} \|\hat{\mu}_\alpha\|_\infty$ ($\leqq \|\mu\|$).

LEMMA 1. *For* $f \in C(G)$

$$\|f\|_A = \sup \left\{ \left| \int_G f d\mu \right| \colon \mu \in M(G), \|\hat{\mu}\|_\infty \leqq 1 \right\}.$$

LEMMA 2.

$$\|\hat{\mu}\|_\infty = \sup\{\|f * \mu\|_2 \colon f \in L^2(G), \|f\|_2 \leqq 1\}$$

($* \equiv$ *convolution*).

2. **Induced representations.** (For reference see R. Godement [2].) Let H be a closed subgroup of G (then $\hat{H}$ is defined similarly to $\hat{G}$, and τ_i is an element of the class $i \in \hat{H}$, with character ϕ_i and dimension d_i). For $\alpha \in \hat{G}$, T_α restricted to H is a representation of H hence it is a direct sum of irreducible representations of H, say $T_\alpha(h) = \sum \oplus_{i \in \hat{H}} n_\alpha(i) \tau_i(h)$ for all $h \in H$ where $n_\alpha(i)$ is the number of times that $T_\alpha|_H$ contains τ_i; for a fixed α, $n_\alpha(i) > 0$ for only a finite number of $i \in \hat{H}$ since $d_\alpha = \sum_{i \in \hat{H}} n_\alpha(i) d_i$.

For $i \in \hat{H}$, let ϕ_i' be the element of $M(G)$ defined by $\int_G f d\phi_i' = \int_H f(h)\phi_i(h) dh$ for all $f \in C(G)$ (dh = normalized Haar measure on H). ϕ_i' is called the generalized character of the representation of G induced by τ_i. Now for $\alpha \in \hat{G}$, $(\phi_i')_\alpha^\wedge = \int_H T_\alpha(h^{-1})\phi_i(h) dh$, thus $(\phi_i')_\alpha^\wedge \neq 0$ if and only if $n_\alpha(i) > 0$, but $\phi_i' \neq 0$ in $M(G)$; therefore there exists at least one $\alpha \in \hat{G}$ such that $n_\alpha(i) > 0$. Let ρ be the restriction map: $C(G) \to C(H)$, then ρ^* is an injection: $M(H) \to M(G)$.

LEMMA 3. *Let* $\mu \in M(H)$, *then* $\|\hat{\mu}\|_\infty = \|(\rho^*\mu)^\wedge\|_\infty$.

PROOF.

$$
\begin{aligned}
\|(\rho^*\mu)^\wedge\|_\infty &= \sup_{\alpha \in \hat{G}} \left\| \int_H T_\alpha(h^{-1}) d\mu(h) \right\|_\infty \\
&= \sup_{\alpha \in \hat{G}} \left\| \sum_{i \in \hat{H}} \oplus n_\alpha(i) \hat{\mu}_i \right\|_\infty \\
&= \sup_{\alpha \in \hat{G}} \max_{n_\alpha(i) > 0} \|\hat{\mu}_i\|_\infty = \sup_{i \in \hat{H}} \|\hat{\mu}_i\|_\infty = \|\hat{\mu}\|_\infty. \quad \text{Q.E.D.}
\end{aligned}
$$

3. Main theorems.

THEOREM 1. $A(G)|_H = A(H)$; *if* $F \in A(G)$ *then* $\rho F \in A(H)$ *and* $\|\rho F\|_A \leqq \|F\|_A$, *and for each* [*real*] $f \in A(H)$ *there exists* [*real*] $F \in A(G)$ *such that* $\rho F = f$ *and* $\|F\|_A = \|f\|_A$.

PROOF. Let $F \in A(G)$, then

$$
\begin{aligned}
\|\rho F\|_A &= \sup \left\{ \left| \int_H F d\mu \right| : \mu \in M(H), \|\hat{\mu}\|_\infty \leqq 1 \right\} \\
&\hspace{20em} \text{(by Lemma 1)} \\
&= \sup \left\{ \left| \int_G F d(\rho^*\mu) \right| : \mu \in M(H), \|(\rho^*\mu)^\wedge\|_\infty \leqq 1 \right\} \\
&\hspace{20em} \text{(by Lemma 3)} \\
&\leqq \sup \left\{ \left| \int_G F d\lambda \right| : \lambda \in M(G), \|\hat{\lambda}\|_\infty \leqq 1 \right\} = \|F\|_A.
\end{aligned}
$$

Therefore $A(G)|_H \subset A(H)$, and $\|\rho F\|_A \leqq \|F\|_A$. Conversely, for each $i \in \hat{H}$ choose some $\alpha(i) \in \hat{G}$ such that $n_{\alpha(i)}(i) > 0$, and $\alpha(\bar{i}) = \bar{\alpha}(i)$ (where $\bar{\alpha}$ is the class of $x \to \overline{T}_\alpha(x)$). This is possible (not necessarily uniquely) by the remarks in §2. If $f \in A(H)$ then

$$
f(h) = \sum_{i \in \hat{H}} d_i \operatorname{Tr}(\hat{f}_i \tau_i(h))
$$

for all $h \in H$, thus it suffices to consider f of the form $d_i \operatorname{Tr}(\hat{f}_i \tau_i(h))$, some $i \in \hat{H}$. Let $\alpha = \alpha(i)$ and let $\hat{F}_\alpha = (d_i/d_\alpha)(\hat{f}_i \oplus 0)$ where the direct sum is taken in such a way that $d_\alpha \operatorname{Tr}(\hat{F}_\alpha T_\alpha(h)) = d_i \operatorname{Tr}(\hat{f}_i \tau_i(h))$, all $h \in H$. Then let $F_i(x) = d_\alpha \operatorname{Tr}(\hat{F}_\alpha T_\alpha(x))$, all $x \in G$, thus $\rho F_i = f$ and $\|F_i\|_A = d_\alpha \|\hat{F}_\alpha\|_1 = d_i \|\hat{f}_i\|_1 = \|f\|_A$. For general $f \in A(H)$, there exists for each $i \in \hat{H}$ a function $F_i \in A(G)$ such that $F_i(h) = d_i \operatorname{Tr}(\hat{f}_i \tau_i(h))$ and $\|F_i\|_A = d_i \|\hat{f}_i\|_1$. Let $F = \sum_{i \in \hat{H}} F_i$, convergent in $A(G)$ since

$$\|F\|_A \leqq \sum_{i\in\hat{H}} \|F_i\|_1 = \sum_{i\in\hat{H}} d_i\|\hat{f}_i\|_1 = \|f\|_A.$$

But $\rho F = f$, so $\|F\|_A \geqq \|f\|_A$, thus $\|F\|_A = \|f\|_A$. Further if f is real, then so is F, because $\alpha(\bar{\imath}) = \bar{\alpha}(i)$. Q.E.D.

Let $\mathbf{G}$ be the class of compact groups having infinite abelian subgroups, hence compact infinite abelian subgroups (by closure).

THEOREM 2. *If $G\in\mathbf{G}$ then only real-analytic functions operate in $A(G)$, that is, if ψ is defined on a closed convex set E in the complex plane, and $\psi \circ F\in A(G)$ whenever $F\in A(G)$ and range $F\subset E$ then ψ is real-analytic on E.*

PROOF. Let H be an infinite compact abelian subgroup of G. By Lemma 6.6.2, p. 143 (Rudin [8]), for $n>0$,

$$\sup\{\|e^{if}\|_A:\ \text{real } f \in A(H),\ \|f\|_A \leqq n\} = e^n.$$

For any $\epsilon>0$, there exists a real $f\in A(H)$ such that $\|f\|_A\leqq n$, $\|e^{if}\|_A > e^n-\epsilon$. By Theorem 1, there exists real $F\in A(G)$ such that $\rho F=f$ and $\|F\|_A=\|f\|_A\leqq n$. Then $\rho(e^{iF})=e^{if}$, hence $e^n\geqq\|e^{iF}\|_A\geqq\|e^{if}\|_A>e^n-\epsilon$, therefore

$$\sup\{\|e^{iF}\|_A:\ \text{real } F \in A(G),\ \|F\|_A \leqq n\} = e^n.$$

Now the proof of Helson, Kahane, Katznelson and Rudin [4] applies to $A(G)$ to show that only real-analytic functions operate. Q.E.D.

4. **Remarks.** One may want to extend the theory of the Fourier algebra to homogeneous spaces, but there is a slight snag. Here is the situation: Let G be a compact group, H a closed subgroup, $X=G/H$ the homogeneous space of right cosets of H, then let $A(G/H) = \{f\in A(G): f(x)=f(xh),\ \text{all } x\in G,\ h\in H\}$. $A(G/H)$ can be interpreted as a Banach algebra of continuous functions on X with maximal ideal space$\cong X$. Now suppose that another group G_1 also acts transitively on X and $X\cong G_1/H_1$. Is there a natural isomorphism between $A(G/H)$ and $A(G_1/H_1)$? In general, the answer is no, as may be seen in the case of $SO(4)/SO(3)\cong S^3\cong Sp(1)$, (the group of unit quaternions). In the case $G_1\subset G$, $H_1=H\cap G_1$, Theorem 1 shows that $A(G/H)\subset A(G_1/H_1)$.

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