SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is no other outlet.

ON SEMIGROUFS NEAR THE IDENTITY

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In a number of recent papers [1], [2], [3], and [4] the question of the existence of nontrivial semigroups of operators on a normed space "close to the identity" is considered. In this note a simple theorem is proved from which all known results readily follow.

It is a pleasure to thank R. Hirschfeld for mentioning the problem.

THEOREM. Let G be a group and T an endomorphism of G. Suppose there exists a set F of homomorphisms from G into the additive group C such that:

1. \( f(T^n g) = o(n) \) for every \( f \in F, g \in G \).
2. For every \( 0 \neq b \in G \), there is a \( f_b \in F \) with

\[
\limsup_n \frac{1}{n} \left| \sum_{k=0}^{n-1} f_b(T^k b) \right| > 0.
\]

Then \( T g = g \) for all \( g \in G \).

Proof. Let \( b = T g - g \). For \( f \in F \) we get

\[
\frac{1}{n} \sum_{k=0}^{n-1} f(T^k b) = \frac{1}{n} \left( f(T^n g) - f(g) \right)
\]

and thus

\[
\lim \frac{1}{n} \left( \sum_{k=0}^{n-1} f(T^k b) \right) = 0.
\]

By 2 this implies \( b = 0 \). Q.E.D.

COROLLARY 1. Let \( T \) be an additive operator on a locally convex space \( X \), with topological dual \( X' \), such that

1. \( \langle T^n x, x' \rangle = o(n) \) for every \( x \in X, x' \in X' \).
2. For every \( 0 \neq x \in X \) there is an \( x' \in X' \) with \( \langle x, x' \rangle \neq 0 \) and

\[
\liminf_n \left| \frac{1}{n} \sum_{k=0}^{n-1} \langle T^k x, x' \rangle - \langle x, x' \rangle \right| < \left| \langle x, x' \rangle \right|.
\]

Then \( T = I \).

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Proof. We apply the theorem to \( X = G, F = X' \) and \( T \).
We note that it is even not supposed that \( T \) is continuous.

Corollary 2. Let \( T \) be a continuous operator on a normed space \( X \)
such that
1. \( \| T^n \| = o(n) \).
2. \( \lim \inf (1/n) \| \sum_{k=0}^{n-1} (T^k - I) \| < 1 \).

Then \( T = I \).

Proof. Let \( 0 \neq x \in X \). There is a \( 0 \neq x' \in X' \) with \( \langle x, x' \rangle = \| x \| \cdot \| x' \| \).
Then
\[
\lim \inf \left| \frac{1}{n} \sum_{k=0}^{n-1} \langle (T^k - I)x, x' \rangle \right| < \| x \| \cdot \| x' \| = \langle x, x' \rangle
\]
so that Corollary 1 applies.

References


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