ON CLOSED IMAGES OF THE SPACE OF IRRATIONALS

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In connection with his recent joint work with A. H. Stone on quotients of the space of irrationals (cf. [6]), E. Michael asked whether every complete separable metric space is a continuous image of the irrationals under a closed mapping. The present note contains the positive answer to this question. In fact, we actually prove a more general result which does not assume separability. The proof is based on a refinement of a known lemma (see [5, p. 281], and [2] or [4] for the nonseparable case) that every nonempty closed subset of a metric space X of covering dimension 0 is a retract of X. The refinement, which seems to be new even in the separable case, asserts the existence of a closed retraction. Our construction of such a retraction coincides with one given in [9, Theorem 3′, p. 8] by A. H. Stone, who did not, however, observe that it is closed. We include the details for the sake of completeness. I am grateful to E. Michael for some valuable remarks.

LEMMA. *For every nonempty closed subset F of a metrizable space X satisfying* $\dim(X-F)=0$, *there exists a closed continuous function $f: X\to F$ such that $f| F: F\to F$ is the identity on F.*

PROOF. The function f is defined by a modification of the well-known retraction of X to F (see [5, p. 281]).

Let ρ be a metric in the space X and let $U_1, U_2, \cdots$ be a sequence of open-and-closed subsets of X such that

$$F\subset U_i\subset \{x\colon \rho(x,F)<1/i\}$$

(see [1, Theorems 6.2.3 and 6.2.4]). We can suppose, without loss of generality, that the sequence $U_1, U_2, \cdots$ is decreasing. Let $W_1=X-U_2$ and $W_i=U_i-U_{i+1}$ for $i=2, 3, \cdots$. We have $X-F=\bigcup_{i=1}^{\infty} W_i$. For every i the set W_i can be expressed as the union of a family $\mathfrak{F}_i=\{F_s\}_{s\in S_i}$ of disjoint open-and-closed sets whose diameters are smaller than $1/i$ (see [1, Theorem 7.2.3]); we can suppose that $S_i\cap S_j=0$ for $i\neq j$. Let $S=\bigcup_{i=1}^{\infty} S_i$; it is easy to see that the sets $\{F_s\}_{s\in S}$ are open-and-closed, disjoint and satisfy

(1) If $s_n\in S_{i_n}$ and $\lim i_n=\infty$, then $\lim \delta(F_{s_n})=0=\lim \rho(F, F_{s_n})$,

(2) $X-F=\bigcup_{s\in S} F_s$.

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For every integer i, let P_i be a subset of F maximal with respect to the property "$\rho(x, y) \geqq 1/i$ for any $x, y \in P$." Such a subset exists, because the property is of finite character (see [1, p. 21]). Moreover, we have

(3) P_i has no accumulation points,

(4) for every $x \in F$ there exists a $p \in P_i$ such that $\rho(x, p) < 1/i$.

From (4) it follows that, for every i and $s \in S_i$, there exists a point $p_s \in P_i$ satisfying

(5) $\rho(p_s, F_s) < \rho(F, F_s) + 2/i$.

By (2), the formula

$$\begin{aligned} f(x) &= x \quad \text{for } x \in F, \\ &= p_s \quad \text{for } x \in F_s \end{aligned}$$

defines a function $f: X \to F$. The family $\{F_s\}_{s\in S}$ being composed of open-and-closed and disjoint sets, the function f is continuous on $\bigcup_{s\in S} F_s = X - F$. To prove that f is continuous, it is sufficient to show that

(6) If $x \in F$, $x = \lim x_n$, and $x_n \in X - F$, then $x = f(x) = \lim f(x_n)$.

Let $x_n \in F_{s_n}$, where $s_n \in S_{i_n}$. As $x = \lim x_n \in F$, we have $\lim \rho(x_n, F) = 0$ and it follows that $\lim i_n = \infty$. Hence (6) is a consequence of the following

(7) If $x_n \in F_{s_n}$, $s_n \in S_{i_n}$, and $\lim i_n = \infty$, then $\lim \rho(x_n, f(x_n)) = 0$.

To prove (7) let us note that by (5) and the equality $f(x_n) = p_{s_n}$ we have

$$\begin{aligned} \rho(x_n, f(x_n)) &= \rho(x_n, p_{s_n}) \leqq \delta(F_{s_n}) + \rho(p_{s_n}, F_{s_n}) \\ &\leqq \delta(F_{s_n}) + \rho(F, F_{s_n}) + 2/i_n, \end{aligned}$$

which by (1) implies (7).

Now we shall prove that f is a closed mapping. Let A be a closed subset of X and let

$$x \in \overline{f(A)}.$$

As

$$\overline{f(A)} = \overline{f(A \cap F)} \cup \overline{f(A - F)} = \overline{A \cap F} \cup \overline{f(A - F)},$$

we can assume that either

$$x \in \overline{A \cap F}$$

or there exists a sequence $x_1, x_2, \cdots$ of points of $A - F$ such that $x = \lim f(x_n)$. The set $A \cap F$ is closed, so in the first case we have $x \in A \cap F = f(A \cap F) \subset f(A)$. In the second case, let $x_n \in F_{s_n}$, where

$s_n \in S_{i_n}$. We have then $x = \lim p_{s_n}$, and (3) implies that $\lim i_n = \infty$ or else $x = f(x_n)$ for some n. If $x = f(x_n)$, then of course $x \in f(A)$. If $\lim i_n = \infty$, then by (7) we have $x = \lim x_n$. But A is closed, so $x \in A$, and as $x \in F$ we have $f(x) = x$ and in this case also $x \in f(A)$. This shows that $f(A)$ is closed, and that proves the lemma.

Let us note that, as shown by the example of the real line in the plane, not every retract of an arbitrary metric space X is the image of X under a closed retraction.

For every $\mathfrak{m} \geqq \aleph_0$, let $B(\mathfrak{m})$ denote the Baire space of weight $\mathfrak{m}$, i.e. the Cartesian product of $\aleph_0$ copies of the discrete space of cardinality $\mathfrak{m}$. It is well known (see for example [1, Theorem 7.3.9]) that every metrizable space X satisfying the conditions $\dim X \leqq 0$ and $w(X) \leqq \mathfrak{m}$, where $w(X)$ denotes the weight of X, is embeddable in the space $B(\mathfrak{m})$. Looking at the proof of this result one sees that, if X is moreover complete, then its homeomorphic image in $B(\mathfrak{m})$ is closed. Now if S is a subspace of a metrizable space X, then $\dim S \leqq \dim X$ (cf. [1, Theorem 7.3.3]), and clearly $w(S) \leqq w(X)$. It follows that every G_δ-set in $B(\mathfrak{m})$, which must be complete in some metric, is homeomorphic to a closed subset of the space $B(\mathfrak{m})$.

Theorem. *For every nonempty complete metric space X satisfying $w(X) \leqq \mathfrak{m}$, there exists a continuous closed function of the Baire space $B(\mathfrak{m})$ of weight $\mathfrak{m}$ onto X.*

Proof. By a result of Morita (see [7] or [8]), there exists a perfect mapping g (i.e. a closed mapping such that $g^{-1}(x)$ is compact for any $x \in X$) defined on a subspace S of $B(\mathfrak{m})$ onto X. Since an inverse image of an absolute G_δ under a perfect mapping is an absolute G_δ (see [3]), it follows that S is a G_δ-set in $B(\mathfrak{m})$. But by the above remark, S is homeomorphic to a closed subset S' of $B(\mathfrak{m})$, so we can assume that g is defined on a closed subset S' of $B(\mathfrak{m})$. Composing the retraction of $B(\mathfrak{m})$ onto S' satisfying our lemma with g, we get a closed function of $B(\mathfrak{m})$ onto X.

As $B(\aleph_0)$ is homeomorphic to the space of irrationals, we obtain

Corollary. *For every nonempty complete separable metric space X, there exists a continuous closed function of the space of irrationals onto X.*

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