RETRACTIONS ONTO ANR'S[1]

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The following theorem is proved in [1].

THEOREM 3 OF [1]. *Suppose X is a closed subset of a compact n-manifold M such that X is an i-connected ($i=1, 2, \cdots, n-2$) ANR. Then for each point p of a component U of $M-X$ there is a retraction of $U+X-\{p\}$ onto X.*

It was stated that Theorem 3 could be generalized by dropping the condition that M is compact and that if $\overline{U}$ is not compact, it is not even necessary to remove a point from U. The proof of this extended result is elementary but was not included in [1] since it was thought to be beyond the interests of most readers of the Monthly. However, the extended result has applications [2] so it was decided to publish a proof.

RETRACTION THEOREM. *Suppose M is an n-manifold and X is an i-connected ($i=0, 1, 2, \cdots, n-2$) closed subset of M that is an ANR. For each component U of $M-X$ whose closure is not compact, there is a retraction of $U+X$ onto X.*

PROOF. Let N be an open subset of M containing X such that there is a retraction of $\overline{N}$ onto X.

Let $B_1, B_2, \cdots$ be a locally finite collection of topological n-balls in U such that $U\subset\overline{N}+\sum B_i$. Let $p_i\in\text{Int } B_i-\sum \text{Bd } B_j$ and $P=\sum\{p_i\}$. Also, let $N'=\overline{N}-\sum \text{Int } B_i$, and $B_i'=B_i-\sum_{i+1}^{\infty}\text{Int } B_j$.

Let r_0 be a retraction of N' onto X. The retraction r_0 of N' onto X can be extended inductively to a retraction r_i of $N'+\sum_1^i B_j'-P$ onto X. Once r_i is defined, r_{i+1} is extended as follows. Let f_{i+1} be a retraction of $B_{i+1}-P$ onto Bd B_{i+1} and g_{i+1} be a map of Bd B_{i+1} into X that agrees with r_i on Bd $B_{i+1}\cap(N'+\sum_1^i B_j')$. The connectedness of X is used in showing this extension is possible. Then on $B_{i+1}'-P$, $r_{i+1}=g_{i+1}\circ f_{i+1}$. The limit of the r_i's provide a retraction r of $X+U-P$ onto X.

Let $R_1, R_2, \cdots$ be a locally finite collection of mutually exclusive topological rays in U such that each R_i is closed in M and p_i is the end of R_i. One way to get such a collection is to consider an expanding bull's-eye sequence $C_1, C_2, \cdots$ of compact sets as described later

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with the additional condition that each Bd $C_i \cap P = 0$. Then the R_i's may be chosen to move monotonically out through the C_j's so that if $p_i \notin C_j$, then $R_i \cap C_j = 0$ and if $p_i \in C_j$, then $R_i \cap C_j$ is an arc from p_i to a point of Bd C_j at which Bd C_j is locally an $n-1$ manifold.

In each neighborhood of each R_i there is a closed subset E_i of M such that E_i is homeomorphic to closed Euclidean half space and $p_i \in \text{Int } E_i$. To obtain E_i we can consider the discrete sequence $p_i = p_1^i, p_2^i, \cdots$ of points of R_i such that each p_{j+1}^i is further out R_i than p_j^i and each arc $p_j^i p_{j+1}^i$ of R_i lies in a ball in U. Then E_i results from putting a small ball about $p_i = p_1^i$ so that the ball misses $R_i - p_1^i p_2^i$, pushing a feeler from the ball so that the feeler contains p_2^i and is very near $p_1^i p_2^i$ (so near in fact that it misses $R_i - p_1^i p_3^i$), elongating the feeler until it contains p_3^i with the elongation being very close to $p_2^i p_3^i, \cdots$. Hence using the R_i's as guides to pull a boundary point of balls about the ends of the R_i's out to infinity, one finds that there is a locally finite collection $E_1, E_i, \cdots$ of mutually exclusive sets in U such that each E_i is closed in M, each E_i is homeomorphic to closed Euclidean half space, and $p_i \in \text{Int } E_i$. By retracting each of the E_i's onto its boundary, we find that there is a retraction r' of $X+U$ onto $X+U - \sum \text{Int } E_i$.

A retraction promised by the retraction theorem is given by $r \circ r'$.

The retraction theorem applies to manifolds with boundaries. If U intersects Bd M, we do not need to assume that its closure is not compact. It also applies to pseudo manifold and other spaces but we shall not push it to full generality. However, we do describe a sequence of compact sets whose existence is of interest for its own sake.

EXPANDING BULL'S-EYE SEQUENCE. *If M is an n-manifold and X is a closed subset of M that is an ANR, then there is a sequence of compact sets $C_1, C_2, \cdots$ such that*

(1) $\sum C_i = M$,

(2) $C_i \subset \text{Int} C_{i+1}$,

(3) Bd $C_i - X$ *is locally an $n-1$ manifold except at a set of dimension $n-2$, and*

(4) *each component of $(M-X)-C_i$ which is not a component of $M-X$ has a noncompact closure.*

It is easy to get a sequence $C_1, C_2, \cdots$ satisfying conditions (1) and (2) above and such that each C_i is the union of a finite number of balls. These C_i's would then satisfy condition (3). If C_1 does not satisfy condition (4) it could be enlarged by adding the closure of the union of all components of $(M-X)-C_1$ that intersect C_2 and have compact closures. The following lemma implies that this enlargement is compact, where in applying the lemma we are considering $X-C_1$

as an ANR in the manifold $M-C_1$. By enlarging the other C_i's, a sequence satisfying conditions (1), (2), (3), (4) can be obtained.

LEMMA. *Suppose M is a manifold and X is a closed subset of M that is an ANR. Then no compact subset of M intersects infinitely many components of $M-X$.*

PROOF. Assume the lemma is false and p_0 is a point in the compact set such that each neighborhood of p_0 intersects infinitely many components of $M-X$. Suppose the metric of M is locally Euclidean near p_0 and B, B', B'' are balls centered at p_0 with radii 3ϵ, 2ϵ, ϵ, respectively, and having the ordinary Euclidean metric. Let r be a retraction of a neighborhood N of X onto X which does not move any point more than ϵ and V be a component of $(M-X)$ such that $B'\cap V\subset N$ and there is a point $p_1\in V\cap B''$. We are led to the contradiction that there is a map $f: B'\to \text{Bd } B'$ such that $f|\text{Bd } B'$ is homotopic to the identity on Bd B'. (Of course, there is no such map f since there is no retraction of a ball onto its boundary.) To define the impossible map f of B' onto Bd B' we would let r_1 be the projection from p_1 of $B-\{p_1\}$ onto Bd B' and define $f=r_1$ on $B'-V$ and $f=r_1\circ r$ on $B'\cap V$.

REFERENCES

1. R. H. Bing, *Retractions onto spheres*, Amer. Math. Monthly 71 (1964), 482–484.

2. Robert F. Brown, *A fixed point theorem for open Q-acyclic n-manifolds*, (to appear).

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