RETRACTIONS ONTO ANR’S

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The following theorem is proved in [1].

**Theorem 3 of [1].** Suppose $X$ is a closed subset of a compact $n$-manifold $M$ such that $X$ is an $i$-connected $(i = 1, 2, \ldots, n-2)$ ANR. Then for each point $p$ of a component $U$ of $M - X$ there is a retraction of $U + X - \{p\}$ onto $X$.

It was stated that Theorem 3 could be generalized by dropping the condition that $M$ is compact and that if $U$ is not compact, it is not even necessary to remove a point from $U$. The proof of this extended result is elementary but was not included in [1] since it was thought to be beyond the interests of most readers of the Monthly. However, the extended result has applications [2] so it was decided to publish a proof.

**Retraction Theorem.** Suppose $M$ is an $n$-manifold and $X$ is an $i$-connected $(i = 0, 1, 2, \ldots, n-2)$ closed subset of $M$ that is an ANR. For each component $U$ of $M - X$ whose closure is not compact, there is a retraction of $U + X$ onto $X$.

**Proof.** Let $N$ be an open subset of $M$ containing $X$ such that there is a retraction of $N$ onto $X$.

Let $B_1, B_2, \cdots$ be a locally finite collection of topological $n$-balls in $U$ such that $U \subset N + \sum B_i$. Let $p_i \in \text{Int } B_i - \sum \text{Bd } B_j$ and $P = \sum \{p_i\}$. Also, let $N' = N - \sum \text{Int } B_i$, and $B'_i = B_i - \sum_{j=1}^i \text{Int } B_j$.

Let $r_0$ be a retraction of $N'$ onto $X$. The retraction $r_0$ of $N'$ onto $X$ can be extended inductively to a retraction $r_i$ of $N' + \sum B'_i - P$ onto $X$. Once $r_i$ is defined, $r_{i+1}$ is extended as follows. Let $f_{i+1}$ be a retraction of $B'_{i+1} - P$ onto $\text{Bd } B'_{i+1}$ and $g_{i+1}$ be a map of $\text{Bd } B'_{i+1}$ into $X$ that agrees with $r_i$ on $\text{Bd } B'_{i+1} \cap (N' + \sum B'_i)$. The connectedness of $X$ is used in showing this extension is possible. Then on $B'_{i+1} - P$, $r_{i+1} = g_{i+1} \circ f_{i+1}$. The limit of the $r_i$'s provide a retraction $r$ of $X + U - P$ onto $X$.

Let $R_1, R_2, \cdots$ be a locally finite collection of mutually exclusive topological rays in $U$ such that each $R_i$ is closed in $M$ and $p_i$ is the end of $R_i$. One way to get such a collection is to consider an expanding bull's-eye sequence $C_1, C_2, \cdots$ of compact sets as described later.

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with the additional condition that each $\text{Bd } C_i \cap P = 0$. Then the $R_i$'s may be chosen to move monotonically out through the $C_i$'s so that if $p_i \in C_j$, then $R_i \cap C_j = 0$ and if $p_i \in C_j$, then $R_i \cap C_j$ is an arc from $p_i$ to a point of $\text{Bd } C_i$ at which $\text{Bd } C_i$ is locally an $n-1$ manifold.

In each neighborhood of each $R_i$ there is a closed subset $E_i$ of $M$ such that $E_i$ is homeomorphic to closed Euclidean half space and $p_i \in \text{Int } E_i$. To obtain $E_i$ we can consider the discrete sequence $p_i = p_i^1, p_i^2, \ldots$ of points of $R_i$ such that each $p_i^{j+1}$ is further out $R_i$ than $p_i^j$ and each arc $p_i^j p_i^{j+1}$ of $R_i$ lies in a ball in $U$. Then $E_i$ results from putting a small ball about $p_i = p_i^1$ so that the ball misses $R_i - p_i^1 p_i^2$, pushing a feeler from the ball so that the feeler contains $p_i^2$ and is very near $p_i^1 p_i^2$ (so near in fact that it misses $R_i - p_i^1 p_i^2$), elongating the feeler until it contains $p_i^2$ with the elongation being very close to $p_i^2 p_i^3, \ldots$. Hence using the $R_i$'s as guides to pull a boundary point of balls about the ends of the $R_i$'s out to infinity, one finds that there is a locally finite collection $E_1, E_2, \ldots$ of mutually exclusive sets in $U$ such that each $E_i$ is closed in $M$, each $E_i$ is homeomorphic to closed Euclidean half space, and $p_i \in \text{Int } E_i$. By retracting each of the $E_i$'s onto its boundary, we find that there is a retraction $r'$ of $X + U$ onto $X + U - \sum \text{Int } E_i$.

A retraction promised by the retraction theorem is given by $r' \circ r$. The retraction theorem applies to manifolds with boundaries. If $U$ intersects $\text{Bd } M$, we do not need to assume that its closure is not compact. It also applies to pseudo manifold and other spaces but we shall not push it to full generality. However, we do describe a sequence of compact sets whose existence is of interest for its own sake.

**Expanding Bull's-eye Sequence.** If $M$ is an $n$-manifold and $X$ is a closed subset of $M$ that is an ANR, then there is a sequence of compact sets $C_1, C_2, \ldots$ such that

1. $\sum C_i = M$,
2. $C_i \subseteq \text{Int } C_{i+1}$,
3. $\text{Bd } C_i - X$ is locally an $n-1$ manifold except at a set of dimension $n-2$, and
4. each component of $(M-X)-C_i$ which is not a component of $M-X$ has a noncompact closure.

It is easy to get a sequence $C_1, C_2, \ldots$ satisfying conditions (1) and (2) above and such that each $C_i$ is the union of a finite number of balls. These $C_i$'s would then satisfy condition (3). If $C_i$ does not satisfy condition (4) it could be enlarged by adding the closure of the union of all components of $(M-X)-C_i$ that intersect $C_i$ and have compact closures. The following lemma implies that this enlargement is compact, where in applying the lemma we are considering $X-C_i$. 

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as an ANR in the manifold \( M - C_i \). By enlarging the other \( C_i \)'s, a sequence satisfying conditions (1), (2), (3), (4) can be obtained.

**Lemma.** Suppose \( M \) is a manifold and \( X \) is a closed subset of \( M \) that is an ANR. Then no compact subset of \( M \) intersects infinitely many components of \( M - X \).

**Proof.** Assume the lemma is false and \( p_0 \) is a point in the compact set such that each neighborhood of \( p_0 \) intersects infinitely many components of \( M - X \). Suppose the metric of \( M \) is locally Euclidean near \( p_0 \) and \( B, B', B'' \) are balls centered at \( p_0 \) with radii \( 3\epsilon, 2\epsilon, \epsilon \), respectively, and having the ordinary Euclidean metric. Let \( r \) be a retraction of a neighborhood \( N \) of \( X \) onto \( X \) which does not move any point more than \( \epsilon \) and \( V \) be a component of \( (M - X) \) such that \( B' \cap V \subseteq N \) and there is a point \( p_1 \in V \cap B'' \). We are led to the contradiction that there is a map \( f: B' \to \text{Bd } B' \) such that \( f|_{\text{Bd } B'} \) is homotopic to the identity on \( \text{Bd } B' \). (Of course, there is no such map \( f \) since there is no retraction of a ball onto its boundary.) To define the impossible map \( f \) of \( B' \) onto \( \text{Bd } B' \) we would let \( r_1 \) be the projection from \( p_1 \) of \( B - \{p_1\} \) onto \( \text{Bd } B' \) and define \( f = r_1 \) on \( B' - V \) and \( f = r_1 \circ r \) on \( B' \cap V \).

**References**


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