FIBRE BUNDLES AND MEASURE

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In the study of Banach algebra bundles [2], there arises quite naturally the question of integration of bundle sections if the fibre is a vector space and the base space supports a measure. The possibility of such integration follows from the theorem below. It is published separately because of its independent interest and because of its relevance to the following results in topology.

1. (Doyle-Hocking). Every topological closed n-manifold M^n is the union of an open n-cell and a closed $(n-1)$-dimensional subset disjoint from the cell [1].

2. (Brown-Cassler). Every closed topological n-manifold M^n is the continuous image by a map ϕ of the closed n-cell $I^n \equiv [0, 1]\times[0, 1]\times \cdots \times[0, 1]$ (n factors) so that (i) $\phi|\partial I^n$ is a homeomorphism ($\partial I^n = I^n \cap (R^n \backslash I^n)^-$, (ii) $\phi^{-1}\phi(\partial I^n) = \partial I^n$ and (iii) $\dim \phi(\partial I^n) \leqq n-1$ [1].

3. Any (coordinate) bundle $\mathcal{B}$ over the n-sphere S^n is strictly equivalent to a (coordinate) bundle $\mathcal{B}'$ in normal form. In $\mathcal{B}'$ the open covering of S^n consists of two sets V_1 and V_2 each of which is a zone of S^n and such that $V_1 \cap V_2$ is a narrow equatorial band. The width of the band can be made arbitrarily small [4].

Result 2 is regarded as complementary to 1. Results 1 and 3 show that fibre bundles over many manifolds are really fibre bundles over contractible sets united with sets of small dimension or small measure. Such bundles are trivial over the contractible parts of their base spaces. Consequently, each may be exemplified by a coordinate bundle for which the transition functions (coordinate transformations) are constant and equal to the group identity over "most" of the base space.

The theorem of this paper extends the results above. Broadly paraphrased, the theorem says that a fibre bundle over *any* topological space X endowed with a reasonable topological measure may be exemplified by a coordinate bundle whose transition functions are, on most of X, constant and equal to the identity of the group of the bundle. More precisely we have the

Theorem. *Let $\mathcal{E}$ be a fibre bundle with base space X, fibre F, group G, bundle space E and projection $p: E \to X$. Assume that on the σ-ring $\mathcal{S}$*

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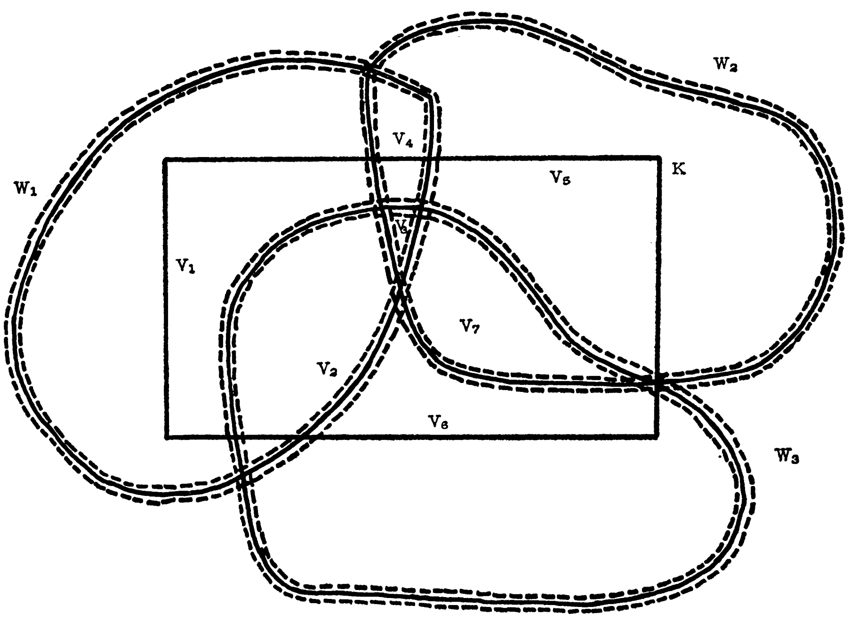
generated by the compact sets of X there is defined a regular measure μ and that X has a neighborhood basis $\mathfrak{N}=\{N\}$ satisfying $\mu(\partial N)=0$ for all $N\in\mathfrak{N}$, where $\partial N=\overline{N}\cap(X\backslash N)^-=$boundary of N. Let K be a compact set in X and let $\epsilon>0$. Then there is a coordinate bundle $\mathcal{B}^0$ in the equivalence class defining $\mathcal{E}$ and a compact set $\tilde{K}\subset K$ such that

1. $\mu(K\backslash\tilde{K})<\epsilon$.

2. *If U', U'' are open sets used in the definition of $\mathcal{B}^0$ and if $x\in U'\cap U''\cap\tilde{K}$, then $g_{U'U''}(x)=e\equiv$identity of G. Here $g_{U'U''}$: $U'\cap U''\to G$ is the transition function or coordinate transformation corresponding to U' and U''* [4].

Remarks. (i) If X is a locally compact group and if μ is Haar measure, then the conditions given above for X and μ are fulfilled [3].

(ii) If a second coordinate bundle $\mathcal{B}^1$ in the equivalence class of $\mathcal{E}$ consists, in particular, of an open covering that is a refinement of the open covering used for $\mathcal{B}^0$ and of maps ϕ_U^1 that are restrictions of the corresponding maps ϕ_U^0 (see below), then $\mathcal{B}^1$ also enjoys properties 1 and 2 above.



Narrow band defined by dashed line is

$$\bigcup_j M_j$$

PROOF. In what follows, if U is an open set used in the definition of a coordinate bundle then by definition,

$$\overset{-1}{\phi_U}: \overset{-1}{p}(U) \to U \times F$$

is the homeomorphism corresponding to local triviality. If $\mathcal{U}$ is an open covering used in the definition of a coordinate bundle $\mathcal{B}^0$, and if $\mathcal{V}$ is a refinement of $\mathcal{U}$ let $\mathcal{V} \ni V \to U(V) \in \mathcal{U}$ be a map such that $V \subset U(V)$. Then we define ϕ_V^{-1} as the restriction to $p^{-1}(V)$ of $\phi_{U(V)}^{-1}$. In this way $\mathcal{V}$ and the maps ϕ_V^{-1} become ingredients of a coordinate bundle $\mathcal{B}^1$ equivalent to $\mathcal{B}^0$ [4]. The idea of the proof can be followed most readily by use of the diagram provided.

Owing to our hypotheses, we may assume we have a coordinate bundle whose covering $\mathcal{W}$ consists of open sets W such that $\mu(\partial W) = 0$. We cover K with finitely many sets $W_1, W_2, \cdots, W_n$ of $\mathcal{W}$. We then choose a refinement $\mathfrak{M} = \{M\}$ of $\mathcal{W}$ so that we may cover $\bigcup_{i=1}^n (\partial W_i)$ by a family $\{M_j\}$ of sets of $\mathfrak{M}$ and so that

$$\mu\left(\bigcup_j M_j\right) < \epsilon.$$

(The regularity of μ makes this possible.) The set $K \backslash (\bigcup_j M_j) \equiv \tilde{K}$ is compact and $\mu(K \backslash \tilde{K}) < \epsilon$.

For each $x \in K \backslash \bigcup_{i=1}^n \partial W_i$, let $S(x) = \{i: x \in W_i\}$. Then let

$$V(x) = \left(\bigcap_{i \in S(x)} W_i\right) \cap \left(\bigcap_{j \notin S(x)} (X \backslash \overline{W}_j)\right).$$

The set $V(x)$ is open. Furthermore, $x \in V(x)$. Otherwise, $x \in \overline{W}_j$ for some $j \notin S(x)$. Since $x \notin W_j$, we see $x \in \partial W_j$, a contradiction. Thus $\bigcup_x V(x) \supset K \backslash \bigcup_{i=1}^n \partial W_i$.

There are only finitely many distinct $V(x)$; let them be $V_1, V_2, \cdots, V_m$. They are pairwise disjoint. Indeed let

$$V_p = \left(\bigcap_{i \in S_p} W_i\right) \cap \left(\bigcap_{j \notin S_p} (X \backslash \overline{W}_j)\right),$$

$$V_q = \left(\bigcap_{i \in S_q} W_i\right) \cap \left(\bigcap_{j \notin S_q} (X \backslash \overline{W}_j)\right).$$

If $S_p = S_q$ then $V_p = V_q$. If $V_p \neq V_q$ then $S_p \neq S_q$, and so there is a j say in S_p and not in S_q. Whence if $x \in V_q$, then $x \in X \backslash \overline{W}_j$ and so $x \notin W_j$, $x \notin V_p$. Hence if $V_p \neq V_q$ then $V_p \cap V_q = \varnothing$. If a V_k meets a W_{i_0}, then

$V_k \subset W_{i_0}$. Indeed, if $V_k = (\bigcap_{i \in S_k} W_i) \cap (\bigcap_{j \notin S_k} (X \setminus \overline{W}_j))$ and if V_k meets W_{i_0}, then $i_0 \in S_k$ and thus $V_k \subset W_{i_0}$.

In summary:

(i) $\bigcup_{k=1}^m V_k \supset K \setminus \bigcup_{i=1}^n \partial W_i$.

(ii) $V_p \cap V_q = \emptyset$ if $p \neq q$.

(iii) Each $V_p \subset$ some W_i and if $V_p \cap W_i \neq \emptyset$, then $V_p \subset W_i$.

Let $\mathfrak{U}$ be the family $\{U\}$ consisting of (a) the open sets $\{M_j\}$; (b) for any M meeting $\tilde{K}$, all nonempty sets $M \cap V_k$; (c) all nonempty sets $M \setminus \tilde{K}$. Then $\mathfrak{U}$ is an open covering. For if $x \in X$, either $x \in \bigcup_j M_j$ or $x \in \tilde{K}$, whence $x \in$ some M and $x \in$ some V_k whence $x \in$ some $M \cap V_k$, or $x \notin \tilde{K}$ and then $x \in$ some M whence $x \in$ some $M \setminus \tilde{K}$. Since each $U \subset$ some M, we see $\mathfrak{U}$ is a refinement of M, hence a refinement of $\mathfrak{W}$.

We well-order $\mathfrak{W}$: $\{W_\lambda : 1 \leqq \lambda < \xi\}$, where ξ is some initial ordinal. Note that the first n elements of $\mathfrak{W}$ are the $W_1, W_2, \cdots, W_n$ above. Then for each $U \in \mathfrak{U}$ let $\lambda(U) = \min\{\lambda : U \subset W_\lambda\}$. This provides a map $\mathfrak{U} \ni U \to W_{\lambda(U)} \in \mathfrak{W}$ and thus permits the definition of a coordinate bundle (a) based on $\mathfrak{U}$, (b) defined via restrictions of the coordinate maps ϕ_W given for $\mathfrak{W}$ and (c) equivalent to the coordinate bundle based on $\mathfrak{W}$.

Consider an $x \in U' \cap U'' \cap \tilde{K}$ where U', $U'' \in \mathfrak{U}$. Since $x \in \tilde{K}$ we see $x \notin \bigcup_j M_j$. Thus neither U' nor U'' is a set M_j nor a set $M_j \cap V_k$ nor a set $M \setminus \tilde{K}$. Thus we see that, for some p and q, $U' = M' \cap V_p$, $U'' = M'' \cap V_q$. Since $x \in U' \cap U'' \neq \emptyset$, we see $V_p \cap V_q \neq \emptyset$, whence $V_p = V_q \equiv V_k$. We assert that $\lambda(U') = \lambda(U'')$. Indeed, clearly $\lambda(U')$, $\lambda(U'') \leqq n$. Furthermore, if $U' \subset W_i$, then V_k meets W_i, $V_k \subset W_i$ (by (iii)) and thus $U'' \subset W_i$. Thus

$$\{\lambda : \lambda \leqq n,\ U' \subset W_\lambda\} = \{\lambda : \lambda \leqq n,\ U'' \subset W_\lambda\},$$

whence $\lambda(U') = \lambda(U'')$. Call their common value r.

It is now clear that

$$\overset{-1}{\phi}_{U'}(\overset{-1}{p}(U')) = \overset{-1}{\phi}_{W_r}(\overset{-1}{p}(U')) \quad \text{and} \quad \overset{-1}{\phi}_{U''}(\overset{-1}{p}(U'')) = \overset{-1}{\phi}_{W_r}(\overset{-1}{p}(U''))$$

whence $g_{U'U''}(x) = e$.

COROLLARY. *If $X \in \mathcal{S}$ and if $\mu(X) < \infty$ then for $\epsilon > 0$ there is an open set $W \in \mathcal{S}$, $\mu(W) < \epsilon$ and, for any refinement of the coordinate bundle indicated in the theorem, $g_{U'U''}(x) = $ identity if $x \in U' \cap U'' \cap (X \setminus W)$.*

The above theorem permits the development of an integration theory in appropriate contexts. We propose to discuss this theory and its application to "group algebra bundles" in a separate paper.

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