ON AN INEQUALITY OF NEHARI

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Nehari [1, Theorem I] claims that if \([a, b]\) contains \(n\) zeros of a nontrivial solution of \(y^{(n)} + p_n y^{(n-1)} + \cdots + p_2 y = 0\), then

\[
\sum_{k=1}^{n} 2^k (b - a)^{n-k} \int_{a}^{b} |p_k|^k > 2^{n+1}.
\]

In a private communication with one of the authors, Professor Nehari has indicated that the inequality is undecided since the argument given in [1] that Theorem II implies Theorem I is incorrect. It is the purpose of this note to show that (1) is correct for \(n = 2\). In fact, we prove a stronger result for

\[
y'' + g y' + f y = 0.
\]

**Theorem.** Let \(a\) and \(b\) be successive zeros of a nontrivial solution to (2) where \(f\) and \(g\) are integrable. Then

\[
(b - a) \int_{a}^{b} f^+(x) dx - 4 \exp\left(-\frac{1}{2} \int_{a}^{b} |g| (x) dx\right) > 0
\]

and a fortiori

\[
(b - a) \int_{a}^{b} f^+(x) dx + 2 \int_{a}^{b} |g| (x) dx > 4.
\]

If \(a\) and \(b\) are successive zeros, then there is a \(c \in (a, b)\) such that \(y'(c) = 0\). Nehari shows that

\[
1 < (c - a) \int_{a}^{c} |f| + \int_{a}^{c} |g|,
\]

and a similar inequality for the interval \((c, b)\). The trick is to combine the two to get (4).

We start with an inequality which is stronger than (5). Consider the equation \((ry')' + p y = 0\) for \(r > 0\), with \(r\) and \(p\) integrable.

**Lemma (See [2]).** If \(y(a) = 0\) and \(y'(c) = 0\), \(a < c\), then

\[
1 < \int_{a}^{c} r^{-1} \int_{a}^{c} p^+.
\]
where \( p^+(x) = \max \{ p(x), 0 \} \).

**Proof.** Let \( y(x) = \max \{ y(t) \} \). Then

\[
(y(x))^2 \leq \left( \int_a^c |y'| \right)^2 = \left( \int_a^c r^{-1/2} |\sqrt{r} y'| \right)^2
\]

\[
< \left( \int_a^c r^{-1} \right) \int_a^c r(y')^2 = \left( \int_a^c r^{-1} \right) \int_a^c p y^2
\]

\[
\leq \left( \int_a^c r^{-1} \right) \int_a^c p^+ y^2 \leq y^2(x) \int_a^c r^{-1} \int_a^c p^+
\]

from which the result follows. We have used the Schwarz inequality and the identity \( \int_a^c r(y')^2 = \int_a^c p y^2 \) which may be verified by an integration by parts.

A similar result holds on \((c, b)\) where \( y(b) = 0 \). Now we want to apply the lemma to the equation (2). Taking \( r = \exp \int_a^c g \) and \( p = \exp \int_a^c g \) we get

\[
(6) \quad 1 < \int_a^c \exp \left( -\int_a^t g(s) ds \right) dt \int_a^c f^+(x) \exp \left( \int_t^c g(s) ds \right) dx.
\]

Writing this as a double integral we see that

\[
1 < \int_a^c \int_a^c \exp \left( \int_t^c g(s) ds \right) f^+(x) dx dt
\]

\[
< \int_a^c \int_a^c \exp \left( \int_t^c g|s| (s) ds \right) f^+(x) dx dt
\]

or

\[
(7) \quad 1 < (c - a) \int_a^c f^+(x) dx \exp \int_a^c |g|(s) ds.
\]

A similar result holds for the interval \((c, b)\) where \( y(b) = 0 \).

In order to motivate a later inequality we observe that (7) implies (5) with \( |f| \) replaced by \( f^+ \). Indeed, letting \( A_0 = \int_a^c |g|(x) dx \) and \( A_1 = (c - a) \int_a^c f^+(x) dx \), this claim is the statement that \( A_1 > \exp(-A_0) \) implies \( A_1 + A_0 > 1 \). This follows from \( e^{-x} + x \geq 1 \) for all \( x \geq 0 \).

Now define \( B_1^2 = (b - c) \int_a^c f^+(x) dx \) and \( B_0 = \int_a^c |g|(x) dx \). Then we have \( B_1^2 > \exp(-B_0) \). Now the inequality that is related to (4) as (7) is to (5) is gotten from 4 \( \exp(-y/2) \geq -2y + 4 \), \( y \geq 0 \).

**Proof of the Theorem.** First we have that
by elementary calculus. In fact, the right member is the minimum of the middle member as function of \( c \). Thus the left-hand side of (3) is greater than

\[
(A_1 + B_1)^2 - 4 \exp\left[-\frac{1}{2}(A_0 + B_0)\right] 
\geq A_1^2 + B_1^2 + 2A_1B_1 - 4 \exp\left[-\frac{1}{2}(A_0 + B_0)\right] 
> \exp(-A_0) + \exp(-B_0) - 2 \exp\left[-\frac{1}{2}(A_0 + B_0)\right] 
= \left[\exp(-\frac{1}{2}A_0) - \exp(-\frac{1}{2}B_0)\right]^2 \geq 0.
\]

This proves (3). Equation (4) follows from the inequality \( 4 \exp(-y/2) + 2y - 4 \geq 0 \) for all \( y \geq 0 \), where \( y \) is replaced by \( \int_a^b |g| \).

We remark that both inequalities (7) and (3) are more enlightening than their counterparts (5) and (4). In particular, they show that \( x = (b-a)f_{df}^+ \) cannot be small unless \( y = \int_a^b |g| \) is very large. In fact, as \( x \to 0 \), \( y \to \infty \). This does not follow from (4). Finally, the inequality (3) is sharp since it reduces to Lyapunov's inequality when \( g = 0 \), and this is known to be sharp, see [3].

**Added in proof.** Professor P. Hartman has pointed out that (3) is announced in Levin, *On linear second order differential equations*, Soviet Math. Dokl. 4 (1963), 1814–1817. The content of this paper is an elementary proof of (3) and the observation that (4) follows from (3).

**References**

2. D. F. St. Mary, *Some oscillation and comparison theorems for \((r(t)y')' + p(t)y = 0\)*, J. Differential Equations (to appear)

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