ON HYPERBOLIC POLYNOMIALS WITH MULTIPLE ROOTS[1]

GIDEON PEYSER

Let $Q(z)\equiv Q(z_1, \cdots, z_n)$ be a polynomial of degree $m\geqq 1$ in $z\equiv(z_1, \cdots, z_n)$ with complex coefficients. Let $P(z)$ be the principal part, homogeneous of degree m. Q is defined to be hyperbolic with respect to the real vector $\xi\equiv(\xi_1, \cdots, \xi_n)\neq 0$, in the sense of Gårding [**1**], if $P(\xi)\neq 0$ and if there exists a real number t_0 such that for $\operatorname{Re}\tau>t_0$, and all real vectors η, $Q(\tau\xi+i\eta)\neq 0$.

The following properties of hyperbolic polynomials are derived in [**1**] and [**2**]. If Q is hyperbolic with respect to ξ, then it is also hyperbolic with respect to $-\xi$. For the principal part P we have

$$P(\tau\xi + i\eta) = P(\xi)\prod_{\nu=1}^{m}(\tau - p_\nu(\xi, i\eta)) \tag{1}$$

where the roots $p_\nu(\xi, i\eta)$ are purely imaginary for all real vectors η.

Q is called strictly hyperbolic with respect to ξ if the imaginary roots of (1) are distinct for all real vectors η not proportional to ξ. The reference to ξ will be omitted from now on, but will always be tacitly assumed. Now, if Q is hyperbolic then $Q+R$ is hyperbolic for all arbitrary R of degree $<m$ if and only if Q is strictly hyperbolic. Gårding [**1**] proved the sufficiency of this criterion (see also [**4**], where the term total hyperbolicity, instead of strict hyperbolicity, is used). Hörmander [**2**], using a theorem by A. Lax [**3**], showed that the criterion is also necessary.

We consider in this paper hyperbolic polynomials Q that are not necessarily strictly hyperbolic. We derive a criterion regarding the degree, say r, such that $Q+S$ is hyperbolic for all arbitrary S of degree $<r$, but not for all S of degree r.

The integer $d\geqq 0$ is defined as the degree of degeneracy of the hyperbolic polynomial Q, if $d+1$ is the highest multiplicity of the roots p_ν of the principal part $P(\tau\xi+i\eta)=0$ for all real η not proportional to ξ. (A strictly hyperbolic polynomial has therefore the degree of degeneracy 0.)

We shall prove

THEOREM 1. *If d is the degree of degeneracy of the hyperbolic polynomial Q, then $Q+S$ is hyperbolic for all arbitrary polynomials S of*

Received by the editors August 14, 1968.

[1] This work was supported by the National Science Foundation grant GP-8601.

degree $<m-d$ *and there exists a polynomial* S_1 *of degree* $m-d$ *such that* $Q+S_1$ *is not hyperbolic.*

This theorem has the following relationship to linear hyperbolic partial differential equations with constant coefficients. Consider the differential operator $Q(\partial/\partial x)\equiv Q(\partial/\partial x_1, \cdots, \partial/\partial x_n)$. The Cauchy problem for $Q(\partial/\partial x)$ is said to be uniformly well posed with respect to ξ if

(a) for every function $g(x)\in C^\infty$, $x\in R^n$, there exists a unique function $u(x)\in C^\infty$, such that $Q(\partial/\partial x)u=0$ and the derivatives of order $<m$ of $u-g$ vanish on the plane $(x, \xi)\equiv(x_1\xi_1+ \cdots +x_n\xi_n)=0$, and

(b) if the sequence $u^{(k)}\in C^\infty$ satisfies $Q(\partial/\partial x)u^{(k)}=0$ and $u^{(k)}$ and every derivative of $u^{(k)}$ converges to zero, as $k\to\infty$, uniformly on every compact set of the plane $(x, \xi)=0$, then $u^{(k)}$ and every derivative of $u^{(k)}$ converges uniformly to zero on every compact set of R^n.

Gårding [1] showed that the Cauchy problem for $Q(\partial/\partial x)$ is uniformly well posed if and only if Q is hyperbolic. This, together with our Theorem 1, results therefore in

THEOREM 2. *If Q is hyperbolic with degree of degeneracy d, then the Cauchy problem for $Q(\partial/\partial x)+S(\partial/\partial x)$ is uniformly well posed for arbitrary S of degree $<m-d$, and there exists S_1 of degree $m-d$, such that the Cauchy problem for $Q(\partial/\partial x)+S_1(\partial/\partial x)$ is not uniformly well posed.*

We now proceed to prove Theorem 1. The given hyperbolic polynomial Q has the degree of degeneracy d. We want to show that for any S of degree $<m-d$, $T=Q+S$ is also hyperbolic, i.e. that there exists a real number t_1 such that for $\operatorname{Re}\tau>t_1$ and any real vector η, $T(\tau\xi+i\eta)\neq 0$. Now, $\eta=c\xi+\eta_1$ where c is a real number and η_1 orthogonal to ξ. Therefore $T(\tau\xi+i\eta)=T((\tau+ic)\xi+i\eta_1)=T(\sigma\xi+i\eta_1)$. If we show that $T(\sigma\xi+i\eta_1)\neq 0$ for $\operatorname{Re}\sigma>t_2$ then also $T(\tau\xi+i\eta)\neq 0$ for $\operatorname{Re}\tau>t_2$. Hence only η orthogonal to ξ have to be considered. Furthermore if $|\eta|\equiv(\eta_1^2+ \cdots +\eta_n^2)^{1/2}<1$ then the roots of $T(\tau\xi+i\eta)=0$ are uniformly bounded. Therefore, for some t_3, $T(\tau\xi+i\eta)\neq 0$ if $\operatorname{Re}\tau>t_3$. Hence we now restrict η to the set, which we denote by H_ξ, of all real vectors η orthogonal to ξ, and with $|\eta|\geqq 1$. In particular, if $\eta\in H_\xi$ then it is not proportional to ξ.

We denote

$$Q(\tau) = Q(\tau\xi + i\eta), \qquad P(\tau) = P(\tau\xi + i\eta),$$
$$S(\tau) = S(\tau\xi + i\eta), \tag{2}$$

and define

$$V(\tau) = \left(\frac{\partial}{\partial\tau}\right)^d Q(\tau), \tag{3}$$

$$W(\tau) = \left(\frac{\partial}{\partial\tau}\right)^d P(\tau), \tag{4}$$

Since Q is hyperbolic with degree of degeneracy d, it follows that the roots $w_\nu = w_\nu(i\eta)$, $\nu = 1, \cdots, m-d$ of $W(\tau)=0$ are imaginary and distinct for all $\eta \in H_\xi$. The homogeneity of W implies that for a suitable labelling of the roots, $w_\nu(i\eta) = |\eta| w_\nu(i\eta/|\eta|)$. From the distinctness of the roots it follows that there exists a positive constant $b>0$, such that for $\nu \neq \mu$ and $\eta \in H_\xi$, $|w_\nu(i\eta/|\eta|) - w_\mu(i\eta/|\eta|)| \geqq b$ and hence

$$| w_\nu(i\eta) - w_\mu(i\eta) | \geqq |\eta| b. \tag{5}$$

We need two lemmas.

LEMMA 1. *If Q is hyperbolic and $S(\tau)$ and $V(\tau)$ are as defined in* (2) *and* (3), *there exists a number t_4 such that if* Re $\tau > t_4$ *then* $|S(\tau)| < |V(\tau)|$ *for all* $\eta \in H_\xi$.

PROOF. $W(\tau) \equiv W(\tau\xi + i\eta) = W(\xi)\prod_{\nu=1}^{m-d}(\tau - w_\nu(i\eta))$ with $W(\xi) \neq 0$. Denote

$$\Delta_\nu(\tau) = W(\tau)/(\tau - w_\nu).$$

From the Lagrange interpolation formula we have for $\eta \in H_\xi$, $S(\tau) = \sum_{\nu=1}^{m-d} \delta_\nu \Delta_\nu$ where $\delta_\nu = S(w_\nu)/\Delta_\nu(w_\nu)$. From (5) it follows that $|\Delta_\nu(w_\nu)| \geqq |W(\xi)| b^{m-d-1} |\eta|^{m-d-1}$. S is of degree $\leqq m-d-1$. Hence, for $|\eta| \geqq 1$, $|S(w_\nu)| \leqq e|\eta|^{m-d-1}$, e constant. Therefore the δ_ν are uniformly bounded for all $\eta \in H_\xi$, $|\delta_\nu| \leqq a$, $\nu = 1, \cdots, m-d$.

Choosing $t_4 = 2a(m-d)$ then, since all w_ν are imaginary, $|\tau - w_\nu| > 2a(m-d)$ for Re $\tau > t_4$, and therefore

$$|S(\tau)| \leqq \sum_{\nu=1}^{m-d} |\delta_\nu| |W(\tau)| |\tau - w_\nu|^{-1} < \tfrac{1}{2} |W(\tau)|. \tag{6}$$

Now, $V(\tau) = W(\tau) + Y(\tau)$ where Y is of degree $\leqq m-d-1$. Replacing in (6) $S(\tau)$ by $Y(\tau)$, we may assume t_4 to be such that also $|Y(\tau)| < \frac{1}{2}|W(\tau)|$ for Re $\tau > t_4$. Therefore $|V(\tau)| \geqq |W(\tau)| - |Y(\tau)| > \frac{1}{2}|W(\tau)| > |S(\tau)|$, which proves Lemma 1.

LEMMA 2. *There exists t_5 such that for* Re $\tau > t_5$ *and all* $\eta \in H_\xi$, $|V(\tau)| \leqq |Q(\tau)|$.

PROOF. If $d=0$ then $V(\tau) = Q(\tau)$ and the lemma is obvious. For $d>0$, $V(\tau)$ is the sum of terms, each of which consists of $m-d$ factors of $Q(\tau)=P(\xi)\prod_{\nu=1}^{m} (\tau-q_\nu(i\eta))$. The number of terms is $k=m(m-1)\cdots(m-d+1)$. Since Q is hyperbolic, the numbers $\operatorname{Re} q_\nu(i\eta)$ are uniformly bounded from above for all real vectors η. We can therefore choose t_5 such that for $\operatorname{Re}\tau>t_5$, $|\tau-q_\nu(i\eta)|>k(\geqq 1)$. Hence each term of $V(\tau)$ is absolutely $\leqq k^{-1}|Q(\tau)|$. Therefore $|V(\tau)|\leqq|Q(\tau)|$ which proves Lemma 2.

If we consider the numbers t_4 and t_5 of Lemmas 1 and 2, and choose $t_1=\max(t_4, t_5)$ then we have for $\operatorname{Re}\tau>t_1$ and all $\eta\in H_\xi$

$$|Q(\tau)+S(\tau)| \geqq |Q(\tau)| - |S(\tau)| \geqq |V(\tau)| - |S(\tau)| > 0.$$

Hence $Q(\tau)+S(\tau)\neq 0$ which concludes the first part of Theorem 1.

To prove the second part of Theorem 1, we have to show the existence of a polynomial S_1 of degree $m-d$ such that $Q+S_1$ is not hyperbolic. This will be derived as a consequence of a theorem by A. Lax [3]. We use the version of this theorem by Hörmander [2, p. 136], which states that if Q is hyperbolic then the degree of $Q(\xi+\tau i\eta)$ with respect to τ, does not exceed that of the principal part $P(\xi+\tau i\eta)$.

Since d is the degree of degeneracy of Q, there exists a real vector η_0, not proportional to ξ, such that $P(\tau\xi+i\eta_0)=0$ has a $(d+1)$-fold root, say τ_0. Then $P(\tau\xi+i\eta')=0$ has the $(d+1)$-fold root 0, where $\eta'=\eta_0-i\tau_0\xi$ which is again a real vector, not proportional to ξ. Therefore the highest power of τ in $P(\xi+\tau i\eta')=\tau^m P(\tau^{-1}\xi+i\eta')$ is $m-d-1$. Consequently, if S_1 is such that the degree of $S_1(\xi+\tau i\eta')$ with respect to τ is $m-d$, then $Q+S_1$ is not hyperbolic. There is at least one nonzero component of η', say $\eta_i'\neq 0$. We define $S_1(z)=z_i^{m-d}$. Then $S_1(\xi+\tau i\eta')=(\xi_i+\tau i\eta_i')^{m-d}$ which is of degree $m-d$ in τ. Hence S_1 has the required property.

References

1. L. Gårding, *Linear hyperbolic partial differential equations with constant coefficients*, Acta Math. 85 (1950), 1–62.

2. L. Hörmander, *Linear partial differential operators*, Academic Press, New York, 1963.

3. A. Lax, *On Cauchy's problem for partial differential equations with multiple characteristics*, Comm. Pure Appl. Math. 9 (1956), 135–169.

4. G. Peyser, *Energy inequalities for hyperbolic equations in several variables with multiple characteristics and constant coefficients*, Trans. Amer. Math. Soc. 108 (1963), 478–490.

Newark College of Engineering