A NOTE ON DERIVATION PAIRS

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1. Introduction. Let \( G \) be a region in the complex plane and \( H(G) \) denote the vector space of functions analytic on \( G \). Let \( L \) and \( M \) be two linear functionals on \( H(G) \). The pair \( \{L, M\} \) is a derivation pair if

\[
L(fg) = L(f)M(g) + L(g)M(f), \quad f, g \in H(G).
\]

The purpose of this paper is to determine all derivation pairs generalising a result of L. A. Rubel [1]. This incidentally answers a question raised by him viz., whether the functionals satisfying (1) are continuous.

We denote by \( I \) the identity function and \( I^2 \) will then denote the function defined by \( I^2(z) = I(z)^2 \). Throughout we assume \( \{L, M\} \) to be a derivation pair and \( L \neq 0 \).

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**Theorem.** Let \( \{L, M\} \) be a derivation pair. Then one of the following is true:

(i) there exists a \( z_1 \in G \) such that

\[
L(f) = L(I)f(z_1), \quad M(f) = \frac{1}{2}f(z_1), \quad f \in H(G);
\]

(ii) there exists a \( z_1 \in G \) such that

\[
L(f) = L(I)f'(z_1), \quad M(f) = f(z_1), \quad f \in H(G);
\]

(iii) there exists \( z_1, z_2 \in G \) (\( z_1 \neq z_2 \)) such that

\[
L(f) = \frac{L(I)}{z_1 - z_2} (f(z_1) - f(z_2)), \quad M(f) = \frac{1}{2}(f(z_1) + f(z_2)), \quad f \in H(G).
\]

2. Lemmas.

**Lemma 1.** If \( N \) is a multiplicative linear functional on \( H(G) \), then there exists a \( z_0 \in G \) such that \( N(f) = f(z_0) \), \( f \in H(G) \).

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The proof of the above lemma is simple and we include it for the sake of completeness.

**Proof.** Since $N$ is multiplicative we have $N(f) = N(f)N(1)$ which implies $N(1) = 1$. Let $N(I) = z_0$ so that $N(I - z_0) = N(I) - z_0 N(1) = 0$. We now claim that this $z_0$ will satisfy our requirement. First we show that $z_0 \in G$. Suppose not. Then $1/(I - z_0) \in H(G)$ and

$$1 = N\left(\frac{1}{I - z_0}\right) = N(1 - z_0)N\left(\frac{1}{I - z_0}\right) = 0$$

which is impossible.

Now let $f \in H(G)$. Consider the analytic function $g$ defined by $(I - z_0)g = f - f(z_0)$. Applying $N$ to the function $(I - z_0)g$ we obtain $0 = N(f) - N(f(z_0))$ or $N(f) = f(z_0)$. This completes the proof of lemma.

**Lemma 2.** If $L(1) \neq 0$, then $L/L(1)$ and $2M$ are multiplicative.

**Proof.** By (1) it follows that $L(1) = 2L(1)M(1)$ so that $M(1) = \frac{1}{2}$. Then again using (1) it is easy to show that

$$M(f) = L(f)/2L(1), \quad f \in H(G).$$

Substituting this value of $M$ in (1) the result follows.

When $L(1) = 0$, we have $L(f) = L(f)M(1)$ and since $L \neq 0$, this implies that $M(1) = 1$.

**Lemma 3.** Let \( \{L, M\} \) be a derivation pair and suppose $L(1) = 0$. Then

(a) $L(I) \neq 0$.

(b) $M$ is multiplicative when $M(I^2) = M(I)^2$.

(c) If $f$ is defined at $z_0$ and $f \in H(G)$, then

$$L(f) = L(I)M\left(\frac{f - f(z_0)}{I - z_0}\right), \quad f \in H(G),$$

where $z_0 = M(I)$.

**Proof.** (a) Suppose $L(I) = 0$. Let $M(I) = z_0$. If $z_0 \in G$, and $f \in H(G)$, define $f(z_0) = 0$. For all $f \in H(G)$, $(f - f(z_0))/(I - z_0) \in H(G)$ so that

$$L(f) = L(f - f(z_0)) = L\left(\frac{f - f(z_0)}{I - z_0}\right) = L(I)M\left(\frac{f - f(z_0)}{I - z_0}\right) = 0, \quad f \in H(G).$$

Hence $L = 0$ which is a contradiction. This proves (a).
(b) Using (1) we obtain on the one hand

\[ L(I^2f) = L(I^2)M(f) + L(f)M(I^2) = 2L(I)M(I)M(f) + L(f)M(I)^2 \]

and on the other,

\[ L(I^2f) = L(I \cdot If) = L(I)M(If) + L(If)M(I) = L(I)M(If) + L(I)M(I)M(f) + L(f)M(I)^2. \]

Comparing the two expressions for \( L(I^2f) \) and noting that \( L(I) \neq 0 \), we get \( M(If) = M(I)M(f), f \in H(G) \). From this relation it is easy to show (as in Lemma 1) that there exists a \( z_0 \in G \) such that \( M(f) = f(z_0), f \in H(G) \). This implies \( M \) is multiplicative.

(c) \( \frac{f-f(z_0)}{(I-z_0)} \in H(G) \) and

\[
L(f) = L(f \cdot f(z_0)) = L\left( \frac{f-f(z_0)}{I-z_0} \right)
\]

\[
= L(I - z_0)M\left( \frac{f-f(z_0)}{I-z_0} \right) + M(I - z_0)L\left( \frac{f-f(z_0)}{I-z_0} \right)
\]

\[
= L(I)M\left( \frac{f-f(z_0)}{I-z_0} \right)
\]

since \( L(1) = 0 \) and \( M(I - z_0) = 0 \).

This completes the proof of lemma.

Now if \( f, g \in H(G) \) and are defined at \( z_0 \), then applying (c) to \( L(fg), L(f) \) and \( L(g) \) and substituting in (1) we get

\[
M\left( \frac{fg-f(z_0)g(z_0)}{I-z_0} \right) = M(f)M\left( \frac{g-g(z_0)}{I-z_0} \right)
\]

\[
+ M(g)M\left( \frac{f-f(z_0)}{I-z_0} \right).
\]

Put \( f = I^2 - z_0I \),

\[
g = 1/(I-z_1) \quad (z_1 \in G, z_1 \neq z_0).
\]

On noting that \( I/(I-z_1) = 1 + z_1/(I-z_1) \), (2) simplifies with these special values to

\[
M\left( \frac{1}{I-z_1} \right) \left\{ M(I^2) - \frac{z_0^2}{z_1 - z_0} + z_0 - z_1 \right\} = 1.
\]

3. Proof of theorem. Case (i). \( L(1) \neq 0 \). Then \( L/L(1) \) is multiplicative by Lemma 2. By Lemma 1, there exists \( z_1 \in G \) such that \( L(f)/L(1) \)

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\[ M(f) = L(f) / 2L(1) = f(z_1) / 2, \quad f \in H(G). \] This gives (i) of our theorem.

Case (ii). \( L(1) = 0 \). We have two possibilities \( M(I^2) = M(I)^2 \) or \( M(I^2) \neq M(I)^2 \).

If \( M(I^2) = M(I)^2 \), then \( M \) is multiplicative by Lemma 3. Apply Lemma 1 to get \( z_1 \in G \) such that \( M(f) = f(z_1), \quad f \in H(G) \). Now we will prove \( L(f) = L(I)f'(z_1) \). Recall that \( M(I) = z_1 \). Since \( z_1 \in G \), it follows that \( f - f(z_1) \in H(G) \) for all \( f \in H(G) \) and then, by (1) and \( L(1) = 0 \),

\[
L(f) = L(f - f(z_1)) = L((I - z_1)(f - f(z_1)) / (I - z_1)) = L(I)M((f - f(z_1)) / (I - z_1)) = L(I)f'(z_1)
\]

and we obtain (ii) of our theorem.

It remains to consider the case when \( M(I^2) \neq M(I)^2 = z_0^2 \) and \( L(I) = 0 \).

Since \( M(I^2) - z_0^2 \neq 0 \), there are two distinct roots, \( z_1 \) and \( z_2 \), say, of

\[
(M(I^2) - z_0^2)(z - z_0) + z_0 - z = 0.
\]

These roots satisfy

\[
z_1 + z_2 = 2z_0, \quad z_1z_2 = 2z_0 - M(I^2),
\]

so that

\[ M\left\{(I - z_1)(I - z_2)\right\} = M(I^2) - (z_1 + z_2)M(I) + z_1z_2M(1) = 0. \]

Also by (1), (4) and \( L(1) = 0 \),

\[ L\left\{(I - z_1)(I - z_2)\right\} = L(I)M(I - z_2) + L(I)M(I - z_1) = 0. \]

\( z_1 \in G \), since otherwise \( M(1 / (I - z_1)) \) would be finite, contradicting (3). Similarly \( z_2 \in G \).

We can now prove that we have case (iii) of the theorem.

Let \( f \in H(G) \). Then

\[
g = \frac{1}{I - z_2} \left[ \frac{f - f(z_1)}{I - z_1} - \frac{f(z_2) - f(z_1)}{2 - z_1} \right] \in H(G).
\]

Applying (1), (4), and (5) to

\[
(I - z_1)(I - z_2)g = \frac{1}{z_2 - z_1} \left[ (z_2 - z_1)(f - f(z_1)) - (I - z_1)(f(z_2) - f(z_1)) \right].
\]

We obtain
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$L(f) = \frac{L(I)}{z_1 - z_2} [f(z_1) - f(z_2)], \quad f \in H(G).$

From the relation $L((I-z_0)f) = L(I)M(f), \quad f \in H(G)$, we obtain $M(f) = \frac{1}{2}(f(z_1) + f(z_2)), \quad f \in H(G).$ The theorem is completely proved.

REFERENCE


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