PURE STATES AND APPROXIMATE IDENTITIES

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1. Introduction. In this note we show that each norm separable $C^*$-algebra has an increasing abelian approximate identity and that, if the algebra has an identity, each pure state is multiplicative on some maximal abelian subalgebra.

Let $A$ be a $C^*$-algebra without an identity element. A net $\{u_i\}_{i \in I} \subseteq A$, where $I$ is a directed index set, is called an approximate identity for $A$ if $\|u_i\| \leq 1$ for all $i \in I$, and $\|u_i x - x\| \to 0$; $\|x u_i - x\| \to 0$ for all $x \in A$. We say that $\{u_i\}_{i \in I}$ is increasing if $u_i = 0$ and $i < j \Rightarrow u_i \leq u_j$ for all $i, j \in I$. With $u_i$ selfadjoint, if one of the limits exists, so does the other. Each $C^*$-algebra has an increasing approximate identity [2, 1.7.2]. An approximate identity $\{u_i\}_{i \in I}$ is countable if $I$ is countable. It is abelian if $u_i$ and $u_j$ commute for all $i, j \in I$.

An element $x \in A$ is said to be strictly positive if $\rho(x) > 0$ for each nonzero positive linear functional $\rho$ on $A$. A strictly positive element is positive [2, 2.6.2].

We use the following notation: If $M$ is a collection of vectors in a Hilbert space $H$, and $\mathcal{F}$ is a family of bounded linear operators on $H$, then $[\mathcal{F}M]$ is the closed linear span of the set \{ $F\xi$: $F \in \mathcal{F}$, $\xi \in M$ \}.

2. Results.

Lemma 1. If $x \in A$ is strictly positive, and $\pi$ is a nondegenerate representation of $A$ on a Hilbert space $H$, then $[\pi(x)H] = H$.

Proof. Suppose $0 \neq \xi \notin [\pi(x)H]^\perp$. Since $\pi$ is nondegenerate there is an element $a \in A$ such that $\pi(a)\xi \neq 0$. Let $\rho = \omega_\xi \circ \pi$, where $\omega_\xi$ is the positive linear functional $y \mapsto (\xi, y\xi)$. Then $\rho(a^* a) = (\pi(a^* a)\xi, \xi) = \|\pi(a)\xi\|^2 \neq 0$, so $\rho$ is a positive, nonzero linear functional on $A$. But $\rho(x) = (\pi(x)\xi, \xi) = 0$ which contradicts the assumption that $x$ is strictly positive. Hence $[\pi(x)H]^\perp = (0)$ and the lemma follows.

Theorem 1. A $C^*$-algebra $A$ has a countable increasing abelian approximate identity if and only if $A$ contains a strictly positive element.

Proof. If $x_0 \in A$ is strictly positive, we may take $x_0$ with norm equal to 1. Let $u_i = x_0^{1/i}$, and observe that $u_i \geq 0$, $\|u_i\| = 1$; $i \geq j \Rightarrow u_i \geq u_j$ and $u_i$ and $u_j$ commute for all $i, j \in I$. We want to show that for any

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Let \( x \geq 0 \), and let \( y \) be the unique positive square root of \( x \). Let \( \hat{A} \) be the C*-algebra obtained by adjoining an identity \( e \) to \( A \). Then \( u_i \leq e \) for all \( i \), so that \( 0 \leq yu_i y \leq yey = x \), and \( yu_i y \leq yu_j y \) if \( i \leq j \). Hence, if \( z_i = x - yu_i y \), \( \{ z_i \} \) becomes a monotone decreasing sequence of positive elements in \( A \). We claim that \( \| z_i \| \to 0 \). Let 

\[ S = \{ \rho \in A^* : \rho \geq 0, \| \rho \| \leq 1 \}. \]

\( S \) is compact in the \( w^* \)-topology [2, 2.5.5]. We may regard each \( z_i \) as a continuous function on \( S \) by the evaluation map. Since \( z_i \geq 0 \), \( \| z_i \| = \sup \{ \rho(z_i) : \rho \in S \} \) [2, 2.7.3]; so that it suffices to show that \( z_i \) converges uniformly to 0 on \( S \). As the sequence \( z_i \) is monotone, this will follow from Dini’s theorem once we know that \( \rho(z_i) \to 0 \) for each \( \rho \in S \).

Let \( \pi \) be a nondegenerate representation of \( A \) on a Hilbert space \( H \). \( \pi(u_i) = \pi(x_0^{1/2}) = \pi(x_0) u_i \), which by spectral theory converges strongly to the range projection of \( \pi(x_0) \). Since \( x_0 \) is strictly positive it follows by Lemma 1 that \( \pi(u_i) \to I \) strongly on \( H \), where \( I \) is the identity operator on \( H \).

Let \( \rho \neq 0 \) be an arbitrary element of \( S \) and \( \pi_\rho \) be the associated representation of \( A \) on the Hilbert space \( H_\rho \). Then \( \pi_\rho \) is nondegenerate with a cyclic vector \( \xi_\rho \) and

\[
\rho(z_i) = (\pi_\rho(z_i) \xi_\rho, \xi_\rho) \\
= ((\pi_\rho(x) - \pi_\rho(yu_i y)) \xi_\rho, \xi_\rho) \\
= (\pi_\rho(y) \xi_\rho - \pi_\rho(u_i) \pi_\rho(y) \xi_\rho, \pi_\rho(y) \xi_\rho)
\]

which converges to zero since \( \pi_\rho(u_i) \to I \) strongly. Hence \( \| z_i \| \to 0 \).

Working in \( \hat{A} \) (as Akemann does in [1]), let \( v_i \) be the positive square root of \( e - u_i \). Then

\[
\| yu_i y \| = \| yv_i y \| = \| y(e - u_i) y \| = \| x - yu_i y \| \to 0,
\]

and hence

\[
\| xu_i - x \| = \| y^* v_i ^* \| \leq \| y \| \cdot \| yv_i \| \cdot \| v_i \| \leq \| y \| \cdot \| yu_i \| \to 0.
\]

Thus \( \{ u_i \} \) is an approximate identity.

Conversely, suppose \( \{ u_i \} \) is an increasing abelian approximate identity, and let \( x = \sum_{n=1}^{\infty} 2^{-n} u_n \). If \( \rho \) is a nonzero positive linear functional on \( A \), we know that \( \rho(u_n) \to \| \rho \| \) [2, 2.1.5]. Hence \( \rho(u_n) > 0 \) for some \( n \), so \( \rho(x) = \sum_{n=1}^{\infty} 2^{-n} \rho(u_n) > 0 \). This shows that \( x \) is strictly positive, and the proof is complete.

Observe that if \( A \) is separable, then \( A \) has a strictly positive element. Indeed, if \( \{ y_n \} \) is dense in \( A \), then \( \{ x_n = y_n^* y_n \} \) is dense in \( A^+ = \{ x \in A : x \geq 0 \} \). Clearly \( x = \sum_{n=1}^{\infty} (2^n \| x_n \|)^{-1} x_n \) is strictly positive in \( A \).
Corollary 1. Any separable C*-algebra has a countable increasing abelian approximate identity.

Remark. Let $X$ be a locally compact Hausdorff space, $A = C^0(X)$, the C*-algebra of all continuous complex functions on $X$ vanishing at infinity. It is easily verified that $A$ contains a function $f$ which is everywhere positive if and only if $X$ is $\sigma$-compact. Since each state on $A$ may be represented by a Borel measure on $X$, we see that such a function $f$ is a strictly positive element of $A$. Evidently $X$ may be $\sigma$-compact without having a countable base for its topology, so $A$ may have a strictly positive element without being separable. Needless to say, $A$ will not always have strictly positive elements. An example is $C^0(R)$, when $R$ is given the discrete topology.

A positive linear functional $\rho$ on a C*-algebra $A$ is a state if $\|\rho\| = 1$. If $A$ has an identity $e$, this is equivalent to the condition $\rho(e) = 1$. We say $\rho$ is pure if $\rho \neq 0$ and each positive, linear functional $\gamma$ on $A$ such that $0 \leq \gamma \leq \rho$, is of the form $\gamma = a\rho; 0 \leq a \leq 1$.

Theorem 2. Let $A$ be a separable C*-algebra with identity. If $\rho$ is a pure state on $A$, then there is a maximal abelian C*-subalgebra $B$ of $A$ such that $\rho|B$ is multiplicative.

Proof. Let $N_\rho$ be $\{x \in A : \rho(x^*x) = 0\}$ and $A_0 = N_\rho \cap N_\rho^*$. $A_0$ is a C*-subalgebra of $A$ and is therefore separable. Hence $A_0$ contains a strictly positive element $x_0$. Let $B_0$ be a maximal abelian C*-subalgebra of $A_0$ containing $x_0$ and $B$ be $B_0 + C\cdot e$. Then $B$ is an abelian C*-subalgebra of $A$. Since $\rho$ vanishes on $B_0$, $\rho|B$ is multiplicative of norm 1. To show that $B$ is a maximal abelian C*-subalgebra of $A$, it suffices to show that if a selfadjoint $x$ in $A$ commutes with $B$, then $x \in B$. Now, $x \in B$ if and only if $x - \rho(x)e \in B$; so we may assume that $\rho(x) = 0$. Let $\pi_\rho$ be the irreducible representation of $A$ associated with $\rho$ on the Hilbert space $H_\rho$, with cyclic vector $\xi_\rho$ [2, 2.5.4]. Let $H_0 = [\xi_\rho]^1$. We claim that $[\pi_\rho(A_0)H_0] = H_0$. Indeed, if $y \in A_0$ then $\|\pi_\rho(y)\xi_\rho\|^2 = (\pi_\rho(y^*y)\xi_\rho, \xi_\rho) = \rho(y^*y) = 0$; so that $(\pi_\rho(y)\pi_\rho(x)\xi_\rho, \xi_\rho) = 0$ for all $x$ in $A$. On the other hand, let $\xi$ in $H_0$ be arbitrary. By the transivity theorem [2, 2.8.3] there is a selfadjoint element $y \in A$ such that $\pi_\rho(y)\xi_\rho = 0$ and $\pi_\rho(y)\xi = \xi$. But then $y \in A_0$ and the claim follows. Hence $\pi_\rho|A_0$ is a nondegenerate representation on $H_0$. Let $E$ be the orthogonal projection of $H_\rho$ onto $H_0$. By Lemma 1, $[\pi_\rho(x_0)H_0] = H_0$. Now $x_0$ and $x$ commute; so that $E$ and $\pi_\rho(x)$ commute. Hence $H_0$ and $[\xi_\rho]$ are invariant under $\pi_\rho(x)$. This means that $\xi_\rho$ is an eigenvector for $\pi_\rho(x)$; so $\pi_\rho(x)\xi_\rho = a\xi_\rho$ for some real $a$. Now $0 = \rho(x) = (\pi_\rho(x)\xi_\rho, \xi_\rho) = a(\xi_\rho, \xi_\rho) = a$. Hence $\pi_\rho(x)\xi_\rho = 0$ so $x \in A_0$. Since $B_0$ is
maximal abelian in $A_0$ it follows that $x\in B_0 \subseteq B$. The proof is complete.

REFERENCES


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