SOME REMARKS ON HYPERSPACES
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1. The purpose of this paper is to answer a question of R. Schori [3] and to provide simpler arguments for some generalizations of Schori's results.

If $X$ is a metric space, the hyperspace of $X$, denoted $2^X$, is the space of all nonvoid closed subsets of $X$ with the usual Hausdorff metric. The $n$-fold ($n \geq 1$) symmetric product (Borsuk-Ulam [1]) of $X$, denoted $X(n)$, is the subspace of $2^X$ consisting of all elements with $\leq n$ points. Let $I$ denote the closed unit interval, $I^n$ the $n$-cube and $I^\infty$ the Hilbert cube. Let $S(X)$ denote the subspace of $2^X$ consisting of all continua. In [3] R. Schori shows that for $\alpha = \infty, 1, 2, \cdots, I^\alpha(n)$ contains $I^\alpha$ as a factor; that is, $I^\alpha(n)$ is homeomorphic to $Y \times I^\alpha$ for some space $Y$. Let $J^\infty$ denote another copy of the Hilbert cube with $J = [-1, 1]$ and let $R$ be the equivalence relation on $J^\infty$ defined by identifying each $x = (x_1, x_2, \cdots)$ with $-x = (-x_1, -x_2, \cdots)$.

Theorem I. $J^\infty/R$ is not homeomorphic to $J^\infty$.

Thus we settle a question of R. Schori [3].

Proof. Let us suppose it were. Consider the natural quotient map $P: J^\infty \rightarrow J^\infty/R$. Evidently the restriction of $P: J^\infty - 0 \rightarrow J^\infty/R - P(0)$ is a two-fold covering. Since the Hilbert cube is homogeneous, it follows from the assumption that $J^\infty/R - P(0)$ is simply connected and therefore (well-known) does not admit a two-fold covering. This is a contradiction.

Question. Is $J^\infty/R$ an Absolute Retract?

The question is interesting because $J^\infty/R$ is clearly a retract of $J^\infty/R \times J^\infty$, which is homeomorphic to $J^\infty(2)$ by [3]. A negative answer would imply that $J^\infty(2)$ is not homeomorphic to $J^\infty$.

Theorem II. Let $m, n$ be positive integers. If $X = I^\infty(n)$, $2(I^\infty)$ or $S(I^\infty)$, then $X$ contains $I^m$ as a factor.

Remark. Schori's proof is restricted to symmetric products since it makes strong use of the following well-known characterization of $I^m(n)$. If $n$ is a positive integer, then $I^m(n)$ is homeomorphic to $I^m/R$ where $R$ is the equivalent relation on $I^m$ defined by $(x_1, \cdots, x_n)$

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$R(y_1, \cdots, y_n)$ iff $\{x_1, \cdots, x_n\} = \{y_1, \cdots, y_n\}$ ($x_i$ and $y_i$ are points in $I^n$). However, as such he is able to include the case when $m = \infty$ in his theorem. On the other hand, by working directly with the subspaces of $2^m$ we are able to give a much simplified proof and although we are not able to include $m = \infty$, we generalize in the direction of more general subspaces of $2^m$ which, as the nature of the technique, may include even more subclasses than those mentioned in Theorem II. In the case when $m = \infty$ we are able to prove the following partial generalization:

THEOREM III. If $X = I^n(n)$, $S(I^n)$ or $2^m$, then for any positive integer $k$, $X$ contains $I^k$ as a factor.

Question. If $X = S(I^n)$ or $2^m$, must $X$ contain $I^\infty$ as a factor?

2. The Cone Lemma. The cone over a space $X$, denoted $C(X)$, is the quotient space of $X \times I$ obtained by identifying $X \times 1$ as a point $v$, where $v$ is called the vertex of $C(X)$. Inductively for $n > 1$, define $C^n(X) = C(C^{n-1}(X))$. Let $\cong$ denote “homeomorphic to”.

LEMMA. (Schori). For $n > 1$, $C^n(X) \cong C(X) \times I^{n-1}$.

Outline of Proof. By induction it suffices to consider $n = 2$; that is, $C^2(X) \cong C(X) \times I$. For each $x \in X$, $C^2(x)$ can be realized as a triangle and thus we can deform $C^2(x)$ into $C(x) \times I$. If we do this uniformly for each $x$ (detail in [3]), we obtain a homeomorphism from $C^2(X)$ onto $C(X) \times I$.

Proof of Theorem II. Let $\{v_i\}$ be the unit points in Euclidean space $E^{m+1}$; that is, $v_i$ has 1 for its $i$th-coordinate and 0 otherwise. Let $\sigma$ denote the $m$-simplex $v_1v_2 \cdots v_{m+1}$. Since $\sigma \cong I^m$, it is clear we can replace $I^m$ by $\sigma$ in Theorem II. For each $i$ let $\sigma_i$ be the $(m-1)$-dimensional face $v_1 \cdots v_{i-1}v_{i+1} \cdots v_{m+1}$. Now let $X_0$ be any space in Theorem II. For $i \geq 1$ let $X_i = \{x \in X_0 | x \cap \sigma_i \neq \emptyset \text{ for all } k \leq i\}$. Clearly $C(X_i+1) = \{t_\sigma i+1 + (1-t)x : t \in I, x \in X_{i+1}\} \subset X_i$. We contend that $\{t_\sigma i+1 + (1-t)x : t \in I, x \in X_{i+1}\} = X_i$. Suppose $x(\neq v_{i+1}) \in X_i$. Let $t = \min \pi i+1(x)$, where $\pi i+1$ is the usual projection map. It is routine to verify that $x' = x/(1-t) - t_\sigma i+1 \in X_{i+1}$ and thus $t_\sigma i+1 + (1-t)x' = x$. Inductively, we have $X_0 \cong C(X_1) \cong C(C(X_2)) \cdots \cong C^{m+1}(X_{m+1})$. The theorem now follows from the Cone Lemma.

Proof of Theorem III. Let $s$ denote the infinite product of reals and let $T = \{(x_1, x_2, \cdots) \in I^\infty | 0 \leq x_i \leq 1 \text{ and } \sum_{i=1}^{\infty} x_i \leq 1\}$. Evidently $T$ is closed in $I^\infty$ and therefore compact. Thus $T$ is a compact convex subset of the locally convex topological linear space $s$ which admits a countable family of continuous linear forms that separate
points (namely, the family \( \{ \pi_i \} \) of projections) and thus by [2] \( T \) is homeomorphic to \( I^n \). Hence we may replace \( I^n \) by \( T \) in Theorem III. Now let \( X_0 \) be any space in Theorem III and let \( k \) be any positive integer. For each \( i \geq 1 \), let \( T_i = \{(x_1, x_2, \ldots) \in T | x_i = 0\} \) and \( X_i = \{x \in X_0 | x \cap T_k \neq \emptyset \text{ for all } k \leq i\} \). As in Theorem II, \( C(X_{i+1}) \approx X_i \). Inductively, \( X_0 \approx C(X_1) \cdots \approx C^{k+1}(X_{k+1}) \). The theorem now follows from the Cone Lemma.

References