STRONG TRANSFINITE ORDINAL DIMENSION[1]

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1. **Introduction.** Strong ordinal dimension is defined inductively by $\text{Ind } X = -1$ iff $X = \emptyset$ and $\text{Ind } X \leqq \alpha$ where α is an ordinal, $\alpha \geqq 0$, if and only if whenever $F \subseteq G$ where F is closed and G open, there exists U open such that $F \subseteq U \subseteq G$ and $\text{Ind } B(U) < \alpha$ where $B(U)$ refers to the boundary of U. Weak inductive dimension is defined by $\text{ind } X = -1$ iff $X = \emptyset$ and $\text{ind } X \leqq \alpha$ iff whenever $x \in G$ where G is open, there exists an open set U such that $x \in U \subseteq G$ and $\text{ind } B(U) < \alpha$. The material in this article parallels the second half of G. H. Toulmin's paper [1] extending some of his results for weak transfinite ordinal dimension to the strong transfinite ordinal dimension. First we require the following definitions:

DEFINITION 1.1. An open collection $\mathfrak{U}$ is a basis for the neighborhoods of closed sets in X iff whenever $F \subseteq G$ where F is closed and G open, there exists $U \in \mathfrak{U}$ such that $F \subseteq U \subseteq G$.

DEFINITION 1.2. An open collection $\mathfrak{U}$ is a basis for the neighborhoods of a particular set F in X iff whenever $F \subseteq G$ where G is open, there exists $U \in \mathfrak{U}$ such that $F \subseteq U \subseteq G$.

DEFINITION 1.3. For any ordinal $\alpha \geqq 0$, $\text{Ind } (X; F) \leqq \alpha$ where F is a closed subset of X iff there exists a basis $\mathfrak{U}$ for the neighborhoods of F in X such that $\text{Ind } B(U) < \alpha$ for every $U \in \mathfrak{U}$. We call $\text{Ind } (X; F)$ the strong inductive dimension of X relative to the closed subset F.

Then we may observe that

THEOREM 1.4. $\text{Ind } X \leqq \alpha$ *for* $\alpha \geqq 0$ *iff there exists an open collection* $\mathfrak{U}$ *that is a basis for the neighborhoods of closed sets in* X *such that* $\text{Ind } B(U) < \alpha$ *for every* $U \in \mathfrak{U}$.

THEOREM 1.5. $\text{Ind } X = \sup \{\text{Ind } (X; F) \mid F \in \mathfrak{F}\}$ *where* $\mathfrak{F}$ *is the collection of closed subsets in* X, *provided either side exists.*

2. **Subset theorems.** The proofs of the following theorems are quite straightforward and involve only minor modifications of the proofs appearing in Toulmin's paper.

THEOREM 2.1. *If* $Y \subseteq X$, *Y closed and* $\text{Ind } X \leqq \alpha$, *then* $\text{Ind } Y \leqq \alpha$.

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THEOREM 2.2. *If* $F\subseteq Y\subseteq X$, *both* F *and* Y *closed in* X *and* Ind $(X; F) \leqq \alpha$, *then* Ind $(Y; F)\leqq\alpha$.

THEOREM 2.3. *If* X *is normal,* $F\subseteq G\subseteq X$, *where* F *is closed and* G *open in* X *and* Ind $(G; F)\leqq\alpha$, *then* Ind $(X; F)\leqq\alpha$. *Thus the strong inductive dimension relative to a closed subset is a local property in a normal space.*

THEOREM 2.4. *If* X *is completely normal and* $F\subseteq Y\subseteq X$ *where* F *is closed in* Y, *then* Ind $(Y; F)\leqq\alpha$ *iff* X *has an open collection* $\mathfrak{U}$ *that is a basis for the neighborhoods of* F *in* X *(although* F *need not be closed in* X*) such that* Ind $(B(U)\cap Y)<\alpha$ *for every* $U\in\mathfrak{U}$.

THEOREM 2.5. *If* Ind $X\geqq\alpha$, *then there exists a closed subset* Y *of* X *such that* Ind $Y=\alpha$.

3. Sum theorems.

THEOREM 3.1. *If* $X=A\cup C$ *where* A *and* C *are separated (i.e.* $(A^-\cap C)\cup(A\cap C^-)=\emptyset$*) and* Ind $A\leqq\alpha$, Ind $C\leqq\alpha$, *then* Ind $X\leqq\alpha$.

PROOF. The proof is by induction on α. The theorem holds for $\alpha=-1$ so assume its truth for Ind $<\alpha$. Suppose $F\subseteq G$ where F is closed and G is open. Now $F\cap A\subseteq G\cap A$ so there exists U open in A and hence open in X such that $F\cap A\subseteq U\subseteq G\cap A$ and Ind $B(U; A)<\alpha$ where Ind $B(U; A)$ refers to the boundary of U in A. Since A is closed as well as open in X, $B(U; A)=B(U)$. In the same way there exists V open in X such that $F\cap C\subseteq V\subseteq G\cap C$ and Ind $B(V; C)<\alpha$ where $B(V; C)=B(V)$. Defining $S=B(U)\cup B(V)$ we see that this is the union of sets separated in the relative topology of S and hence, invoking the induction hypothesis, Ind $S<\alpha$. Defining $W=U\cup V$ we see that W is open, $F\subseteq W\subseteq G$ and $B(W)=S$ so we may conclude that Ind $X\leqq\alpha$.

The remaining theorems in this section have to do with completely normal spaces and the following lemmas will prove useful.

LEMMA 3.2. *If* X *is completely normal and* $F\subseteq G\subseteq X$, $Y\subseteq X$ *where* F *and* Y *are closed in* X *and* G *is open in* X *and if* $F\cap Y\subseteq V'\subseteq G\cap Y$ *where* V' *is open in* Y, *then there exists* V *open in* X *such that* $F\subseteq V\subseteq G$, $V\cap Y=V'$ *and* $B(V'; Y)=Y\cap B(V)$.

LEMMA 3.3. *If* X *is completely normal and* $V'\subseteq A\subseteq X$ *where* V' *is open in* A, *then there exists* V *open in* X *such that* $V\cap A=V'$ *and* $B(V'; A)=A\cap B(V)$.

LEMMA 3.4. *If* X *is completely normal and* $A\subseteq X$, Ind $A\leqq\alpha$ *and*

$F \subseteq G$ where F is closed in X and G is open in X, then there exists U open in X such that $F \subseteq U \subseteq G$ and Ind $(B(U) \cap A) < \alpha$.

PROOF. By normality, there exists W open so $F \subseteq W \subseteq W^- \subseteq G$. Since Ind $A \leqq \alpha$ we can find V' open in A such that $W^- \cap A \subseteq V' \subseteq G \cap A$ and Ind $B(V'; A) < \alpha$. By Lemma 3.3 there exists V open in X such that $V \cap A = V'$ and $B(V'; A) = A \cap B(V)$. Define $U = (V \cap G) \cup W$ so U is open and $F \subseteq U \subseteq G$. Further it can be verified that $A \cap B(U) \subseteq A \cap B(V)$, so by Theorem 2.1, Ind $(B(U) \cap A) < \alpha$.

Before proceeding it is necessary to define Toulmin's lower sum and upper sum for ordinals. (For the algebra of ordinals, see Chapter 4 of *Set theory*, by F. Hausdorff, Chelsea, New York, 1957.) A shuffling of ordinals α and β is an interleaving of disjoint well-ordered sets representing α and β whose induced order agrees with their original order. The lower sum $\alpha \pm \beta$ is the minimal ordinal shuffling α and β and the upper sum $\alpha \pm \beta$ is the sup of the ordinals shuffling α and β. Both lower and upper sums are commutative in their arguments and $\alpha \pm 0 = \alpha \mp 0 = \alpha$ for all ordinals α. Toulmin observes that $\alpha \mp \beta$ is strictly increasing in both arguments and that if $h(\alpha, \beta)$ is a function also satisfying this property, then $\alpha \mp \beta \leqq h(\alpha, \beta)$ for all ordinals α and β.

Any ordinal α may be written uniquely as $\alpha = \alpha_1 + \alpha_2$ where α_1 is a limit ordinal or zero and α_2 is finite. Toulmin shows that

$$\begin{aligned} \alpha \pm \beta &= \alpha \qquad \text{if } \alpha_1 > \beta_1, \\ &= \beta \qquad \text{if } \beta_1 > \alpha_1, \\ &= \alpha + \beta_2 \qquad \text{if } \alpha_1 = \beta_1. \end{aligned}$$

Also Toulmin defines a function

$$\begin{aligned} f(\alpha, \beta) &= -1 + \{(\alpha + 1) \pm (\beta + 1)\} - 1, \\ &= \max(\alpha, \beta) \qquad \text{if } \alpha_1 \neq \beta_1, \\ &= \alpha + \beta \qquad \text{if } \alpha_1 = \beta_1 = 0, \\ &= \alpha + \beta_2 + 1 \qquad \text{if } \alpha_1 = \beta_1 \neq 0, \end{aligned}$$

which satisfies the following:

(i) $f(\alpha, 0) = f(0, \alpha) = \alpha$ for all α.

(ii) For any $\alpha, \beta > 0$ either $f(\alpha', \beta) < f(\alpha, \beta)$ for all $\alpha' < \alpha$ or $f(\alpha, \beta') < f(\alpha, \beta)$ for all $\beta' < \beta$.

(iii) If $g(\alpha, \beta)$ satisfies (*i*) and (ii), then $f(\alpha, \beta) \leqq g(\alpha, \beta)$ for all α and β.

It is also clear that f is symmetric in its two arguments and nondecreasing in either argument.

Now we may extend two more of Toulmin's results.

THEOREM 3.5. *If X is completely normal, $X=A\cup B$ and* Ind $A=\alpha$, Ind $B=\beta$, *then* $1+\text{Ind } X\leqq(1+\alpha)\mp(1+\beta)$.

PROOF. Supposing the theorem false, choose A and B with α minimal such that the theorem fails. There must then exist F closed in X such that

$$1 + \text{Ind}(X; F) \nleqq (1 + \alpha) \mp (1 + \beta).$$

Suppose G is an arbitrary open set containing F. Then by Lemma 3.4 there exists U open in X such that $F\subseteq U\subseteq G$ and Ind $(B(U)\cap A)=\alpha'<\alpha$. Since $B(U)\cap B$ is a closed subset of B, Ind $(B(U)\cap B)=\beta'\leqq\beta$ by Theorem 2.1. Now $B(U)=(B(U)\cap A)\cup(B(U)\cap B)$, so since $B(U)$ is completely normal and by the assumed minimality of α, $1+\text{Ind}B(U)\leqq(1+\alpha')\mp(1+\beta')<(1+\alpha)\mp(1+\beta)$ since $\mp$ is strictly increasing in both arguments. We may conclude then that $1+\text{Ind }(X; F)\leqq(1+\alpha)\mp(1+\beta)$ which is the desired contradiction.

The next theorem gives a better bound on the strong inductive dimension of the union of two sets with known dimension, but requires that both sets be closed.

THEOREM 3.6. *If X is completely normal and $X=C\cup D$ where C and D are both closed and* Ind $C=\gamma$, Ind $D=\delta$, *then* Ind $X\leqq\alpha\pm(\beta+1)$ *where* $\alpha=\max(\gamma,\delta)$ *and* $\beta=\text{Ind }(C\cap D)$.

PROOF. Note that $C\cap D$ is a closed subset of both C and D, so it does possess strong transfinite dimension and $\beta\leqq\min(\gamma,\delta)\leqq\alpha$. Toulmin shows that since $\beta\leqq\alpha$, the assertion of the theorem is equivalent to proving that

$$1 + \text{Ind } X \leqq f(1 + \beta, 1 + \alpha)$$

where f is as described earlier. Supposing this result untrue, we may select C and D so as to make the stated inequality false with first β minimal and then α minimal for the chosen β. Note $\beta\neq-1$, for this would imply that C and D are separated and the desired result then follows from Theorem 3.1.

For the chosen C and D there must exist a closed set F such that $1+\text{Ind }(X; F)\nleqq f(1+\beta, 1+\alpha)$. Suppose $F\subseteq G$ where G is open. By the properties of f, we know that either $f(1+\beta', 1+\alpha')<f(1+\beta, 1+\alpha)$ for all $\beta'<\beta$, $\alpha'\leqq\alpha$ or $f(1+\beta', 1+\alpha')<f(1+\beta, 1+\alpha)$ for all $\beta'\leqq\beta$, $\alpha'<\alpha$. Further by the known symmetry of f, if $\alpha=\beta$ then both of the above inequalities hold.

Case 1. $f(1+\beta', 1+\alpha')<f(1+\beta, 1+\alpha)$ for all $\beta'<\beta$, $\alpha'\leqq\alpha$. Defining $E=C\cap D$, we have $F\cap E\subseteq G\cap E$ so since Ind $E=\beta$ there exists U' open in E such that $F\cap E\subseteq U'\subseteq G\cap E$ and Ind $B(U'; E)<\beta$. But E is closed and X completely normal, so by Lemma 3.2 we can find U open in X such that $F\subseteq U\subseteq G$, $U\cap E=U'$ and $B(U'; E)=B(U)\cap E$. Define $C'=B(U)\cap C$ and $D'=B(U)\cap D$. Then C' and D' are closed in their union which is itself completely normal, and further Ind $C'\leqq\gamma$, Ind $D'\leqq\delta$ which implies that $\alpha'=\max$ (Ind C', Ind D') $\leqq\alpha$. Also $\beta'=$ Ind $(C'\cap D')=$ Ind $(B(U)\cap E)<\beta$. Then by the assumed minimality of β, we see that

$$1 + \text{Ind}\,(C' \cup D') \leqq f(1 + \beta', 1 + \alpha') < f(1 + \beta, 1 + \alpha)$$

by the assumed property of f. But $C'\cup D'=B(U)$ so we have $1+\text{Ind}B(U)<f(1+\beta, 1+\alpha)$ which implies that $1+\text{Ind}(X; F)\leqq f(1+\beta, 1+\alpha)$. This is the desired contradiction.

Case 2. $f(1+\beta', 1+\alpha')<f(1+\beta, 1+\alpha)$ for all $\beta'\leqq\beta$, $\alpha'<\alpha$ and we can assume $\beta<\alpha$ for if $\beta=\alpha$, Case 1 applies. Now $F\cap C\subseteq G\cap C$ and Ind $C=\gamma$, so there exists U' open in C such that $F\cap C\subseteq U'\subseteq G\cap C$ and Ind $B(U'; C)=\gamma'<\gamma$. Invoking Lemma 3.2 we can find U open in X such that $F\subseteq U\subseteq G$, $U\cap C=U'$ and $B(U'; C)=B(U)\cap C$. By normality there exists H open in X so that $F\subseteq H\subseteq H^-\subseteq U$. Then exactly as above we can find V open in X such that $F\subseteq V\subseteq H$, $V\cap D=V'$, Ind $B(V'; D)=\delta'<\delta$ and $B(V'; D)=B(V)\cap D$. Since $V^-\subseteq H^-\subseteq U$, $B(U)$ and $B(V)$ are disjoint. Now define $C''=(B(U)\cap C)\cup(B(V)\cap D)$ and $D''=C\cap D$. Since $B(U)\cap C$ and $B(V)\cap D$ are separated, we may use Theorem 3.1 to conclude that $\gamma''=\text{Ind } C''\leqq\max(\gamma', \delta')<\alpha$. Also Ind $D''=\beta<\alpha$ so $\alpha''=\max$ (Ind C'', Ind D'')$<\alpha$. Finally $\beta''=$ Ind $(C''\cap D'')\leqq$ Ind $D''=\beta$ so by the minimality of α and β and since C'' and D'' are both closed in their union which is itself completely normal,

$$1 + \text{Ind}\,(C'' \cup D'') \leqq f(1 + \beta'', 1 + \alpha'') < f(1 + \beta, 1 + \alpha)$$

by the assumed property of f. Next define $W=$ Interior of $(U\cap C)\cup(V\cap D)$. Then W is open and we can verify that $F\subseteq W\subseteq G$ and also $B(W)\subseteq C''\cup D''$. Finally by Theorem 2.1,

$$1 + \text{Ind}\,B(W) < f(1 + \beta, 1 + \alpha)$$

which implies $1+\text{Ind}\,(X; F)\leqq f(1+\beta, 1+\alpha)$. This again is the desired contradiction, proving the theorem.

COROLLARY 3.7. *If X is completely normal, $X=A\cup B$, A and B closed in X and* Ind $A=\alpha$, Ind $B=\beta$, *then*

$$1 + \text{Ind } X \leqq f(1 + \alpha, 1 + \beta).$$

COROLLARY 3.8. *If X is completely normal, $X = A \cup B$, A and B closed in X and* $\text{Ind } A = \alpha = \alpha_1 + \alpha_2$ *and* $\text{Ind } B = \beta = \beta_1 + \beta_2$ *where α_1 and β_1 are limit ordinals or zero and α_2 and β_2 are finite, then*

$$\begin{aligned} \text{Ind } X &= \alpha \qquad \text{if } \alpha_1 > \beta_1, \\ \text{Ind } X &= \beta \qquad \text{if } \beta_1 > \alpha_1, \\ \max(\alpha, \beta) \leqq \text{Ind } X &\leqq \alpha + \beta_2 + 1 \qquad \text{if } \alpha_1 = \beta_1. \end{aligned}$$

As a consequence of Corollary 3.8, if X is completely normal and the union of closed subsets A and B and if $\text{Ind } A$, $\text{Ind } B$ exist and $\max(\text{Ind } A, \text{Ind } B) = \alpha$, then $\alpha \leqq \text{Ind } X < \alpha + \omega$. Further if α is a limit ordinal, we may write $\alpha \leqq \text{Ind } X \leqq \alpha + 1$.

Last it should be noted that the results just obtained can be improved upon. Levshenko [2] has proven using a different approach that if X is completely normal, $X = A \cup B$, $\text{Ind } A \leqq \alpha = \alpha_1 + \alpha_2$, $\text{Ind } B \leqq \beta = \beta_1 + \beta_2$ with α_1, α_2, β_1, β_2 as defined previously, then X has strong transfinite ordinal dimension and

$$\begin{aligned} \text{Ind } X &\leqq \alpha \qquad \text{if } \alpha_1 > \beta_1, \\ \text{Ind } X &\leqq \beta \qquad \text{if } \beta_1 > \alpha_1, \\ \text{Ind } X &\leqq \alpha + \beta_2 + 1 \qquad \text{if } \alpha_1 = \beta_1. \end{aligned}$$

This result drops the requirement that A and B be closed in X.

REFERENCES

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