1. Introduction. Strong ordinal dimension is defined inductively by Ind $X = -1$ iff $X = \emptyset$ and Ind $X \leq \alpha$ where $\alpha$ is an ordinal, $\alpha \geq 0$, if and only if whenever $F \subseteq G$ where $F$ is closed and $G$ open, there exists $U$ open such that $F \subseteq U \subseteq G$ and Ind $B(U) < \alpha$ where $B(U)$ refers to the boundary of $U$. Weak inductive dimension is defined by ind $X = -1$ iff $X = \emptyset$ and ind $X \leq \alpha$ iff whenever $x \in G$ where $G$ is open, there exists an open set $U$ such that $x \in U \subseteq G$ and ind $B(U) < \alpha$. The material in this article parallels the second half of G. H. Toulmin's paper [1] extending some of his results for weak transfinite ordinal dimension to the strong transfinite ordinal dimension. First we require the following definitions:

Definition 1.1. An open collection $\mathcal{U}$ is a basis for the neighborhoods of closed sets in $X$ iff whenever $F \subseteq G$ where $F$ is closed and $G$ open, there exists $U \in \mathcal{U}$ such that $F \subseteq U \subseteq G$.

Definition 1.2. An open collection $\mathcal{U}$ is a basis for the neighborhoods of a particular set $F$ in $X$ iff whenever $F \subseteq G$ where $G$ is open, there exists $U \in \mathcal{U}$ such that $F \subseteq U \subseteq G$.

Definition 1.3. For any ordinal $\alpha \geq 0$, Ind $(X; F) \leq \alpha$ where $F$ is a closed subset of $X$ iff there exists a basis $\mathcal{U}$ for the neighborhoods of $F$ in $X$ such that Ind $B(U) < \alpha$ for every $U \in \mathcal{U}$. We call Ind $(X; F)$ the strong inductive dimension of $X$ relative to the closed subset $F$.

Then we may observe that

Theorem 1.4. Ind $X \leq \alpha$ for $\alpha \geq 0$ iff there exists an open collection $\mathcal{U}$ that is a basis for the neighborhoods of closed sets in $X$ such that Ind $B(U) < \alpha$ for every $U \in \mathcal{U}$.

Theorem 1.5. Ind $X = \sup \{\text{Ind } (X; F) \mid F \in \mathcal{F}\}$ where $\mathcal{F}$ is the collection of closed subsets in $X$, provided either side exists.

2. Subset theorems. The proofs of the following theorems are quite straightforward and involve only minor modifications of the proofs appearing in Toulmin's paper.

Theorem 2.1. If $Y \subseteq X$, $Y$ closed and Ind $X \leq \alpha$, then Ind $Y \leq \alpha$. 
Theorem 2.2. If $F \subseteq Y \subseteq X$, both $F$ and $Y$ closed in $X$ and $\text{Ind} (X; F) \leq \alpha$, then $\text{Ind} (Y; F) \leq \alpha$.

Theorem 2.3. If $X$ is normal, $F \subseteq G \subseteq X$, where $F$ is closed and $G$ open in $X$ and $\text{Ind} (G; F) \leq \alpha$, then $\text{Ind} (X; F) \leq \alpha$. Thus the strong inductive dimension relative to a closed subset is a local property in a normal space.

Theorem 2.4. If $X$ is completely normal and $F \subseteq Y \subseteq X$ where $F$ is closed in $Y$, then $\text{Ind} (Y; F) \leq \alpha$ if $X$ has an open collection $\mathcal{U}$ that is a basis for the neighborhoods of $F$ in $X$ (although $F$ need not be closed in $X$) such that $\text{Ind} (B(U) \cap Y) < \alpha$ for every $U \in \mathcal{U}$.

Theorem 2.5. If $\text{Ind} X \geq \alpha$, then there exists a closed subset $Y$ of $X$ such that $\text{Ind} Y = \alpha$.

3. Sum theorems.

Theorem 3.1. If $X = A \cup C$ where $A$ and $C$ are separated (i.e. $(A-C \cap C') \cup (A \cap C') = \emptyset$) and $\text{Ind} A \leq \alpha$, $\text{Ind} C \leq \alpha$, then $\text{Ind} X \leq \alpha$.

Proof. The proof is by induction on $\alpha$. The theorem holds for $\alpha = -1$ so assume its truth for $\text{Ind} < \alpha$. Suppose $F \subseteq G$ where $F$ is closed and $G$ is open. Now $F \cap A \subseteq G \cap A$ so there exists $U$ open in $A$ and hence open in $X$ such that $F \cap A \subseteq U \subseteq G \cap A$ and $\text{Ind} B(U; A) < \alpha$ where $\text{Ind} B(U; A)$ refers to the boundary of $U$ in $A$. Since $A$ is closed as well as open in $X$, $B(U; A) = B(U)$. In the same way there exists $V$ open in $X$ such that $F \cap C \subseteq V \subseteq G \cap C$ and $\text{Ind} B(V; C) < \alpha$ where $B(V; C) = B(V)$. Defining $S = B(U) \cup B(V)$ we see that this is the union of sets separated in the relative topology of $S$ and hence, invoking the induction hypothesis, $\text{Ind} S < \alpha$. Defining $W = U \cup V$ we see that $W$ is open, $F \subseteq W \subseteq G$ and $B(W) = S$ so we may conclude that $\text{Ind} X \leq \alpha$.

The remaining theorems in this section have to do with completely normal spaces and the following lemmas will prove useful.

Lemma 3.2. If $X$ is completely normal and $F \subseteq G \subseteq X$, $Y \subseteq X$ where $F$ and $Y$ are closed in $X$ and $G$ is open in $X$ and if $F \cap Y \subseteq V' \subseteq G \cap Y$ where $V'$ is open in $Y$, then there exists $V$ open in $X$ such that $F \subseteq V \subseteq G$, $V \cap Y = V'$ and $B(V'; Y) = Y \cap B(V)$.

Lemma 3.3. If $X$ is completely normal and $V' \subseteq A \subseteq X$ where $V'$ is open in $A$, then there exists $V$ open in $X$ such that $V \cap A = V'$ and $B(V'; A) = A \cap B(V)$.

Lemma 3.4. If $X$ is completely normal and $A \subseteq X$, $\text{Ind} A \leq \alpha$ and
If \( F \subseteq G \) where \( F \) is closed in \( X \) and \( G \) is open in \( X \), then there exists \( U \) open in \( X \) such that \( F \subseteq U \subseteq G \) and \( \text{Ind} (B(U) \cap A) < \alpha \).

**Proof.** By normality, there exists \( W \) open so \( F \subseteq W \subseteq W' \subseteq G \). Since \( \text{Ind} A \leq \alpha \) we can find \( V' \) open in \( A \) such that \( W' \cap A \subseteq V' \subseteq G \cap A \) and \( \text{Ind} B(V'; A) < \alpha \). By Lemma 3.3 there exists \( V \) open in \( X \) such that \( V \cap A = V' \) and \( B(V'; A) = A \cap B(V) \). Define \( U = (V \cap G) \cup W \) so \( U \) is open and \( F \subseteq U \subseteq G \). Further it can be verified that \( A \cap B(U) \subseteq A \cap B(V) \), so by Theorem 2.1, \( \text{Ind} (B(U) \cap A) < \alpha \).

Before proceeding it is necessary to define Toulmin's lower sum and upper sum for ordinals. (For the algebra of ordinals, see Chapter 4 of *Set theory*, by F. Hausdorff, Chelsea, New York, 1957.) A shuffling of ordinals \( \alpha \) and \( \beta \) is an interleaving of disjoint well-ordered sets representing \( \alpha \) and \( \beta \) whose induced order agrees with their original order. The lower sum \( \alpha \pm \beta \) is the minimal ordinal shuffling \( \alpha \) and \( \beta \) and the upper sum \( \alpha \pm \beta \) is the sup of the ordinals shuffling \( \alpha \) and \( \beta \). Both lower and upper sums are commutative in their arguments and \( \alpha \pm 0 = \alpha \pm 0 = \alpha \) for all ordinals \( \alpha \). Toulmin observes that \( \alpha \pm \beta \) is strictly increasing in both arguments and that if \( h(\alpha, \beta) \) is a function also satisfying this property, then \( \alpha \pm \beta \leq h(\alpha, \beta) \) for all ordinals \( \alpha \) and \( \beta \).

Any ordinal \( \alpha \) may be written uniquely as \( \alpha = \alpha_1 + \alpha_2 \) where \( \alpha_1 \) is a limit ordinal or zero and \( \alpha_2 \) is finite. Toulmin shows that

\[
\alpha \pm \beta = \begin{cases} 
\alpha & \text{if } \alpha_1 > \beta_1, \\
\beta & \text{if } \beta_1 > \alpha_1, \\
\alpha + \beta_2 & \text{if } \alpha_1 = \beta_1.
\end{cases}
\]

Also Toulmin defines a function

\[
f(\alpha, \beta) = \begin{cases} 
-1 + \{ (\alpha + 1) \pm (\beta + 1) \} - 1, & \text{if } \alpha_1 \neq \beta_1, \\
\max(\alpha, \beta) & \text{if } \alpha_1 = \beta_1 = 0, \\
\alpha + \beta_2 + 1 & \text{if } \alpha_1 = \beta_1 \neq 0,
\end{cases}
\]

which satisfies the following:

(i) \( f(\alpha, 0) = f(0, \alpha) = \alpha \) for all \( \alpha \).

(ii) For any \( \alpha, \beta > 0 \) either \( f(\alpha', \beta) < f(\alpha, \beta) \) for all \( \alpha' < \alpha \) or \( f(\alpha, \beta') < f(\alpha, \beta) \) for all \( \beta' < \beta \).

(iii) If \( g(\alpha, \beta) \) satisfies (i) and (ii), then \( f(\alpha, \beta) \leq g(\alpha, \beta) \) for all \( \alpha \) and \( \beta \).

It is also clear that \( f \) is symmetric in its two arguments and non-decreasing in either argument.
Now we may extend two more of Toulmin's results.

**Theorem 3.5. If** $X$ **is completely normal,** $X = A \cup B$ **and** $\text{Ind } A = \alpha$, $\text{Ind } B = \beta$, **then** $1 + \text{Ind } X \leq (1 + \alpha) \supseteq (1 + \beta)$.

**Proof.** Supposing the theorem false, choose $A$ and $B$ with $\alpha$ minimal such that the theorem fails. There must then exist $F$ closed in $X$ such that

$$1 + \text{Ind}(X; F) \leq (1 + \alpha) \supseteq (1 + \beta).$$

Suppose $G$ is an arbitrary open set containing $F$. Then by Lemma 3.4 there exists $U$ open in $X$ such that $F \subseteq U \subseteq G$ and $\text{Ind } (B(U) \cap A) = \alpha' < \alpha$. Since $B(U) \cap B$ is a closed subset of $B$, $\text{Ind } (B(U) \cap B) = \beta' \leq \beta$ by Theorem 2.1. Now $B(U) = (B(U) \cap A) \cup (B(U) \cap B)$, so since $B(U)$ is completely normal and by the assumed minimality of $\alpha$, $1 + \text{Ind} B(U) \leq (1 + \alpha') \supseteq (1 + \beta') < (1 + \alpha) \supseteq (1 + \beta)$ since $\supseteq$ is strictly increasing in both arguments. We may conclude then that $1 + \text{Ind } (X; F) \leq (1 + \alpha) \supseteq (1 + \beta)$ which is the desired contradiction.

The next theorem gives a better bound on the strong inductive dimension of the union of two sets with known dimension, but requires that both sets be closed.

**Theorem 3.6. If** $X$ **is completely normal and** $X = C \cup D$ **where** $C$ **and** $D$ **are both closed and** $\text{Ind } C = \gamma$, $\text{Ind } D = \delta$, **then** $\text{Ind } X \leq \alpha + (\beta + 1)$ **where** $\alpha = \max (\gamma, \delta)$ **and** $\beta = \text{Ind } (C \cap D)$.

**Proof.** Note that $C \cap D$ is a closed subset of both $C$ and $D$, so it does possess strong transfinite dimension and $\beta \leq \min (\gamma, \delta) \leq \alpha$. Toulmin shows that since $\beta \leq \alpha$, the assertion of the theorem is equivalent to proving that

$$1 + \text{Ind } X \leq f(1 + \beta, 1 + \alpha)$$

where $f$ is as described earlier. Supposing this result untrue, we may select $C$ and $D$ so as to make the stated inequality false with first $\beta$ minimal and then $\alpha$ minimal for the chosen $\beta$. Note $\beta \neq -1$, for this would imply that $C$ and $D$ are separated and the desired result then follows from Theorem 3.1.

For the chosen $C$ and $D$ there must exist a closed set $F$ such that $1 + \text{Ind } (X; F) \leq f(1 + \beta, 1 + \alpha) + f(1 + \beta', 1 + \alpha')$ for all $\beta' < \beta$, $\alpha' \leq \alpha$ or $f(1 + \beta', 1 + \alpha') < f(1 + \beta, 1 + \alpha)$. Further by the known symmetry of $f$, if $\alpha = \beta$ then both of the above inequalities hold.
Case 1. \( f(1 + \beta', 1 + \alpha') < f(1 + \beta, 1 + \alpha) \) for all \( \beta' < \beta, \alpha' \leq \alpha \). Defining \( E = C \cap D \), we have \( F \cap E \subseteq G \cap E \) so since \( \text{Ind} E = \beta \) there exists \( U' \) open in \( E \) such that \( F \cap E \subseteq U' \subseteq G \cap E \) and \( \text{Ind} B(U'; E) < \beta \). But \( E \) is closed and \( X \) completely normal, so by Lemma 3.2 we can find \( U \) open in \( X \) such that \( F \subseteq U' \subseteq G, \ U \cap E = U'' \) and \( B(U''; E) = B(U) \cap E \). Define \( C' = B(U) \cap C \) and \( D' = B(U) \cap D \). Then \( C' \) and \( D' \) are closed in their union which is itself completely normal, and further \( \text{Ind} C' \leq \gamma, \text{Ind} D' \leq \delta \) which implies that \( \alpha' = \max (\text{Ind} C', \text{Ind} D') \leq \alpha \). Also \( \beta' = \text{Ind} (C' \cap D') = \text{Ind} (B(U) \cap E) < \beta \). Then by the assumed minimality of \( \beta \), we see that

\[
1 + \text{Ind} (C' \cup D') \leq f(1 + \beta', 1 + \alpha') < f(1 + \beta, 1 + \alpha)
\]

by the assumed property of \( f \). But \( C' \cup D' = B(U) \) so we have \( 1 + \text{Ind} B(U) < f(1 + \beta, 1 + \alpha) \) which implies that \( 1 + \text{Ind} (X; F) \leq f(1 + \beta, 1 + \alpha) \). This is the desired contradiction.

Case 2. \( f(1 + \beta', 1 + \alpha') < f(1 + \beta, 1 + \alpha) \) for all \( \beta' \leq \beta, \alpha' < \alpha \) and we can assume \( \beta < \alpha \) for if \( \beta = \alpha \), Case 1 applies. Now \( F \cap C \subseteq G \cap C \) and \( \text{Ind} C = \gamma \), so there exists \( U'' \) open in \( C \) such that \( F \cap C \subseteq U'' \subseteq G \cap C \) and \( \text{Ind} B(U''; C) = \gamma' < \gamma \). Invoking Lemma 3.2 we can find \( V \) open in \( X \) such that \( F \subseteq U' \subseteq G, \ U \cap C = U' \) and \( B(U''; C) = B(U) \cap C \). By normality there exists \( H \) open in \( X \) so that \( F \subseteq H \subseteq H'' \subseteq U \). Then exactly as above we can find \( V \) open in \( X \) such that \( F \subseteq V' \subseteq H, \ V \cap D = V'' \), \( \text{Ind} B(V''; D) = \delta' < \delta \) and \( B(V''; D) = B(V) \cap D \). Since \( V'' \subseteq H'' \subseteq U \), \( B(U) \) and \( B(V) \) are disjoint. Now define \( C'' = (B(U) \cap C) \cup (B(V) \cap D) \) and \( D'' = C \cap D \). Since \( B(U) \cap C \) and \( B(V) \cap D \) are separated, we may use Theorem 3.1 to conclude that \( \gamma'' = \text{Ind} C'' \leq \max (\gamma', \delta') < \alpha \). Also \( \alpha'' = \text{Ind} D'' = \beta < \alpha \) so \( \alpha'' = \max (\text{Ind} C'', \text{Ind} D'') < \alpha \). Finally \( \beta'' = \text{Ind} (C'' \cap D'') \leq \text{Ind} D'' = \beta \) so by the minimality of \( \alpha \) and \( \beta \) and since \( C'' \) and \( D'' \) are both closed in their union which is itself completely normal,

\[
1 + \text{Ind} (C'' \cup D'') \leq f(1 + \beta'', 1 + \alpha'') < f(1 + \beta, 1 + \alpha)
\]

by the assumed property of \( f \). Next define \( W = \text{Interior of} (U \cap C) \cup (V \cap D) \). Then \( W \) is open and we can verify that \( F \subseteq W \subseteq G \) and also \( B(W) \subseteq C'' \cup D'' \). Finally by Theorem 2.1,

\[
1 + \text{Ind} B(W) < f(1 + \beta, 1 + \alpha)
\]

which implies \( 1 + \text{Ind} (X; F) \leq f(1 + \beta, 1 + \alpha) \). This again is the desired contradiction, proving the theorem.

**Corollary 3.7.** If \( X \) is completely normal, \( X = A \cup B \), \( A \) and \( B \) closed in \( X \) and \( \text{Ind} A = \alpha, \text{Ind} B = \beta \), then
1 + Ind $X \leq f(1 + \alpha, 1 + \beta)$.

**Corollary 3.8.** If $X$ is completely normal, $X = A \cup B$, $A$ and $B$ closed in $X$ and Ind $A = \alpha = \alpha_1 + \alpha_2$ and Ind $B = \beta = \beta_1 + \beta_2$ where $\alpha_1$ and $\beta_1$ are limit ordinals or zero and $\alpha_2$ and $\beta_2$ are finite, then

$$\begin{align*}
\text{Ind } X &= \alpha \quad \text{if } \alpha_1 > \beta_1, \\
\text{Ind } X &= \beta \quad \text{if } \beta_1 > \alpha_1, \\
\max(\alpha, \beta) &\leq \text{Ind } X \leq \alpha_1 \beta_1 + 1 \quad \text{if } \alpha_1 = \beta_1.
\end{align*}$$

As a consequence of Corollary 3.8, if $X$ is completely normal and the union of closed subsets $A$ and $B$ and if Ind $A$, Ind $B$ exist and $\max(\text{Ind } A, \text{Ind } B) = \alpha$, then $\alpha \leq \text{Ind } X < \alpha + \omega$. Further if $\alpha$ is a limit ordinal, we may write $\alpha \leq \text{Ind } X \leq \alpha + 1$.

Last it should be noted that the results just obtained can be improved upon. Levshenko [2] has proven using a different approach that if $X$ is completely normal, $X = A \cup B$, Ind $A \leq \alpha = \alpha_1 + \alpha_2$, Ind $B = \beta = \beta_1 + \beta_2$ with $\alpha_1$, $\alpha_2$, $\beta_1$, $\beta_2$ as defined previously, then $X$ has strong transfinite ordinal dimension and

$$\begin{align*}
\text{Ind } X &\leq \alpha \quad \text{if } \alpha_1 > \beta_1, \\
\text{Ind } X &\leq \beta \quad \text{if } \beta_1 > \alpha_1, \\
\text{Ind } X &\leq \alpha + \beta_2 + 1 \quad \text{if } \alpha_1 = \beta_1.
\end{align*}$$

This result drops the requirement that $A$ and $B$ be closed in $X$.

**References**