EIGENFUNCTION EXPANSIONS OF ANALYTIC FUNCTIONS

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In [5, Theorem 10.2], there was derived a simple result characterizing $C^\infty$ sections $f$ of a vector bundle over a compact manifold, in terms of the rate of decay of the coefficients of $f$ in eigenfunctions of a $C^\infty$ differential operator. Here we derive a similar result for analytic sections, mentioned in [5]. Following the proof are several applications (the first of which motivates the general proof) and an alternate proof based on a conversation with F. E. Browder.

**Theorem.** Let $E$ be a complex vector bundle over the compact real-analytic manifold $X$. Suppose $X$ has an analytic volume element, that $E$ has an analytic Hermitian inner product, and that $A$ is an analytic, elliptic, normal differential operator of order $m$ on the sections of $E$. Let $\{\phi_k\}$ and $\{\lambda_k\}$ be respectively the eigensections and eigenvalues of $A$: $A\phi_k = \lambda_k \phi_k$, and let $\mu_k$ be the positive $m$th root of $|\lambda_k|$. Then $f = \sum f_k \phi_k$ is analytic if and only if the sequence $\{s^{\mu_k}|f_k|\}$ is bounded for some $s > 1$.

The condition of the theorem is equivalent to: $\sum s^{\mu_k}|f_k|^2 < \infty$ for some $s > 1$, as the proof shows.

By normality of $A$ we mean $A^*A = AA^*$. This guarantees the existence of a basis of orthonormal eigensections, as follows. The null space of $A$ is finite dimensional [5, Theorem 8.3], and if $P$ is orthogonal projection onto this null space, then $P + A$ is normal and has trivial null space and closed range. It follows that $P + A$ is an isomorphism from $H^m(E)$ (the space of sections of $E$ all of whose derivatives of order less than $m + 1$ are square integrable) onto $H^0(E)$, the space of square integrable sections of $E$. Then $P + A$ has an inverse $B$ which is a compact normal operator on $H^0(E)$. Since $B$ has orthonormal eigensections $\{\phi_k\}$ with eigenvalues converging to zero, the eigenvalues $\{\lambda_k\}$ of $A$ converge to infinity. More precisely we have

$$\sum |\lambda_k|^{-2n} < \infty,$$

where $n$ is the dimension of $X$. For $|\lambda_k|^{2n}$ are the eigenvalues of

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EIGENFUNCTION EXPANSIONS OF ANALYTIC FUNCTIONS 735

\[(AA^*)^n = A^n(A^*)^n, \text{ while } [P + (AA^*)^n]^{-1} \text{ is an operator of trace class } [5, \text{ Lemma 10.1}].\]

Since the \( \phi_k \) are eigensections of \( A^*A + I \) with eigenvalues \( |\lambda_k|^{2+1} \), we may assume that \( A \) is positive, that \( \lambda_k > 0 \), and that the order \( m \) of \( A \) is even.

The proof depends on imbedding \( X \) in the open manifold \( X' = X \times I \), where \( I \) is the open interval \((0, 2)\), and \( X \) is identified with \( X \times \{1\} \). We rely on the Cauchy-Kowalewski theorem to derive the rate of decay of the coefficients from the analyticity of \( f \), and on the analyticity of solutions of elliptic equations for the converse proof.

Extend the bundle \( E \) in the obvious way to \( X' \), denoting the extension by \( E' = E \times I \). If \( \pi \) is the projection of \( E \) onto \( X \), then \( \pi' : E' \to X' \) is defined by \( \pi'(e, t) = (\pi(e), t) \). We consider sections of \( E' \) as maps \( f' : X \times I \to E \) such that \( \pi f'(x, t) = x \). Define the operator \( A' \) on sections of \( E' \) by \( A f'(x, t) = (Af)(x, t) + i(\partial f / \partial t)^m f(x, t) \), where \( m \) is the order of \( A \). Then \( A' \) is an analytic differential operator on sections of \( E' \), and since we have assumed \( m \) is even and the characteristic polynomial (symbol) of \( A \) is positive definite, it follows that \( A' \) is elliptic.

Suppose now \( f \) is an analytic section of \( E \). Then for some \( \epsilon > 0 \), there is an analytic solution \( f' \) in \( X \times [1, 1+\epsilon] \subseteq X' \) of the Cauchy problem:

\[ A f' = 0, f'(x, 1) = f(x), (\partial f / \partial t)^m f'(x, 1) = 0 \text{ for } j = 1, \ldots, m-1. \]

Writing \( a_k(t) = \int_X \langle f'(x, t), \phi_k(x) \rangle dx \), where \( \langle , \rangle \) denotes the Hermitian inner product in any fibre of \( E \), we have

\[ (i\partial / \partial t)^m a_k(t) = i\int_X \langle Af'(x, t), \phi_k(x) \rangle dx = i\lambda_k a_k(t). \]

Thus \( a_k(t) = \sum_{j=1}^m A_{k,j}(\theta_j)^p \), where \( \mu_k > 0 \), \( (\mu_k)^m = \lambda_k \), and \( \{\theta_j\}_1^m \) are the roots of \( \theta^m = i = \sqrt{-1} \). Applying the data for \( t = 1 \), we find

\[ \sum_{j=1}^m A_{k,j}(\theta_j)^p = \delta_{0,p} f_k, \]

where \( f_k = \int_X \langle f(x), \phi_k(x) \rangle dx \). Thus \( A_{k,j} = c_j \delta_k \), where \( \{c_j\} \) is the unique solution of

\[ \sum_{j=1}^m c_j(\theta_j)^p = \delta_{0,p}, \quad p = 0, \ldots, m - 1. \]

Note that the \( c_j \)'s are quotients of nonvanishing van der Monde determinants and thus no \( c_j = 0 \). Now

\[ \int_X \langle f'(x, t), f'(x, t) \rangle dx = \sum_{k=1}^\infty \left| \sum_{j=1}^m c_j \delta_{k,j} \theta_j^{\mu_k} \right|^2 < \infty. \]
for $1 \leq t < 1 + \epsilon$, so \(| \sum_{i=1}^{m} c_i e^{it\mu_i} | f_k | \) is bounded for some fixed $t > 1$.

Letting $\theta_i = e^{it_1/2m}$, we have $\text{Re}(\theta_i) > \text{Re}(\theta_j)$ for $j = 2, \ldots, m$. Since

$$\sum_{i=1}^{m} c_i e^{it_1\mu_i} = e^{it_1\mu_k} \left( c_1 + \sum_{i=2}^{m} c_i (e^{it_1\mu_i} - e^{it_1\mu_k}) \right),$$

while $\text{Re}(\mu_k (\theta_j - \theta_i)) \to -\infty$ and $c_k \neq 0$, we have that \(| e^{it\mu_k} f_k | \) is bounded. If $\alpha$ is the real part of $\theta_i$, and $s = e^\alpha$, we then have $s > 1$ and \(| s^\alpha | f_k | \) is bounded, which proves the first part of the theorem.

For the converse, we construct an $L^2$ solution $u$ of $A'u = 0$ with $u(x, 1) = f(x)$, and then observe that since $A'$ is analytic and elliptic, $u$ is analytic [2, §5]. The construction of $u$ proceeds as follows.

First, from the boundedness of \(| s^\alpha | f_k | \) we conclude that $\sum t^{2\mu_k} | f_k |^2 < \infty$ for $0 \leq t < s$. For if $r = t/s$, $\sum t^{2\mu_k} | f_k |^2 \leq M \sum r^{2\mu_k}$.

Since $\sum (\mu_k)^{-p} < \infty$ for an appropriate $p$ (by (1)), $\sum | \log r^{2\mu_k} |^{-p} < \infty$, and the comparison test shows that $\sum r^{2\mu_k} < \infty$.

Thus writing $u(x, t) = \sum_{i=1}^{m} \sum_{k} f_{i} c_{j} e^{it\mu_i} \phi_k(x)$ for $s^{-1} < t < s$ (with $c_j$ as in (2) and $(\theta_j)^m = 1$), we have that $u$ is square integrable on every compact subset of $X \times \{ s^{-1} < t < s \}$. It is also easy to show that for each $C^\infty$ section $\psi$ of $E'$ with compact support in $X \times \{ s^{-1} < t < s \}$, we have $(u, (A')^* \psi) = 0$, so that $u$ is a “weak” solution of $A'u = 0$. It follows from standard regularity theorems that $u$ is $C^\infty$ [1, Theorem 8.1], and then analytic [2, §5]. Finally, since $f$ is the restriction of $u$ to $X \times \{ 1 \}$, $f$ is analytic.

**Applications.** If we let $A$ be the Laplace operator on the unit sphere $\{ |x| = 1 \}$ in $R^{n+1}$, then the eigenfunction expansion in question is the spherical harmonic expansion $f(x) = \sum f_{j k} S_{j k}(x)$ ($|x| = 1$) where $S_{j k}$ is a spherical harmonic of degree $j$. The eigenvalues are $\lambda_{j k} = -j(j+n-2)$, and $k$ runs from 1 to $(2j+n-2)(j+n-3)!/j!(n-2)!$.

Thus it follows easily from the general theorem above that $f$ is analytic if and only if $\sum f_{j k} r^j S_{j k}$ converges (in $L^2$) for some $r > 1$. Let now $\mathcal{H}$ be the space of functions harmonic in $\{ |x| < 1 \}$, with the topology of uniform convergence on compact sets; and let $\mathcal{G}$ be the set of functions analytic on $\{ |x| = 1 \}$, untopologized. Then we can show immediately that $\mathcal{G}$ is the dual of $\mathcal{H}$. For this, use the base of neighborhoods of zero in $\mathcal{H}$ given by

$$U_{r, \delta} = \{ u \in \mathcal{H} : \int_{|x| < 1} |u(rx)|^2 dx < \delta \} \quad \text{for } 0 < r < 1, \ \delta > 0.$$ 

Suppose $f^\wedge$ is in the dual of $\mathcal{H}$, let $H_{j k}(x) = |x|^j S_{j k}(x/|x|)$, and set
Suppose \( f^*(u) \) is defined on \( H_j \), where \( H_j = \{ f : \mathbb{C} \rightarrow \mathbb{C} \mid f \text{ is analytic in } \mathbb{C}_j \} \). Let \( u = \sum u_j H_j \). Then

\[
\sum |u_j| \leq \delta, \quad \sum |f_j r^j| < \delta^{-1},
\]

which shows that the \( f_j \) are the spherical harmonic coefficients of a function \( f \) in \( \mathcal{A} \). Conversely, each function in \( \mathcal{A} \) leads to a functional on \( \mathcal{C} \), and the isomorphism is established. The same isomorphism can also be realized as follows. Given \( f \) analytic on \( \{|x| = 1\} \), solve the problem (i) \( \Delta v(x) = 0 \) in \( |x| > 1 \), (ii) \( v \) bounded in \( |x| > 1 \), (iii) \( v(x) = f(x) \) for \( |x| = 1 \). Then \( v \) extends analytically to \( |x| \leq r \) for some \( r < 1 \), and for any \( u \) in \( \mathcal{C} \) we have \( f^*(u) = \int_{|x|=1} u(rx) v(rx) \). \( \mathcal{A} \) can now be given the various topologies of the dual of \( \mathcal{C} \).

For a more general result of this type, see Lions and Magenes [7].

For a second application, suppose \( f \) is analytic in \( \mathbb{R}^{n+1} - \{0\} \), and for some complex \( \lambda \), \( f(tx) = t^\lambda f(x) \) for all \( t > 0 \). Then (except for certain integer values of \( \lambda \)), \( f \) defines a unique tempered distribution on \( \mathbb{R}^{n+1} \), which has a Fourier transform \( f^\sim \). If \( f(x) = |x|^\lambda \sum f_j S_j(x/|x|) \), then \( f^\sim(x) \) comes from the function \( |x|^{-\lambda-n-1} \sum f_j \gamma_j S_j(x/|x|) \), with \( \gamma_j = \pi^{n/2} (-i)^{2n+1} \Gamma((j+n+\lambda)/2) / \Gamma((j-\lambda)/2) \) (see [4]). Since \( \sum |f_j| t^{2s} < \infty \) for some \( t > 1 \), so is \( \sum |f_j| \gamma_j s^{2s} < \infty \) for some \( s > 1 \), and \( f \) is analytic. The same result holds, with minor rephrasings, for the exceptional integer values mentioned above.

Another corollary of the expansion theorem is the following: if \( B \) is any bounded operator on \( H^0(E) \) and \( AB = BA \), then \( B \) maps analytic functions into analytic functions. For if \( \{\lambda_j\} \) are the distinct eigenvalues of \( A \), and \( S_j \) is the eigenspace of \( \lambda_j \), then any \( f \) in \( H^0(E) \) has the expansion \( f = \sum a_j \phi_j \), where \( \phi_j \in S_j \), and \( \{\phi_j\} \) extends to an orthonormal basis of eigensections. If \( B\phi_j = b_j \psi_j \) with \( b_j \) complex and \( \|\psi_j\| = 1 \), then \( |b_j| \leq \|B\| \), \( \psi_j \in S_j \), and \( \{\psi_j\} \) extends to an orthonormal basis of eigenfunctions. Since \( Bf = \sum a_j b_j \psi_j \), we find \( Bf \) is analytic when \( f \) is.

Finally, if \( A \) is a positive semidefinite elliptic operator, then for each positive number \( \varepsilon \) and each real number \( \alpha \) there is a well defined positive operator \( (A + \varepsilon)^\alpha \) on \( H^0(E) \). It is an easy consequence of the above theorem that, if \( A \) is analytic, then \( (A + \varepsilon)^\alpha \) maps analytic functions into analytic functions, and in fact the map is continuous and invertible in appropriate topologies. In particular, the operator \( \Delta = (L - \Delta)^1/2 \) constructed in [6] has this property.

Alternate proof. We can also base the proof of our main theorem on a result of Kotake and Narasimhan [3]. This result uses the technique of [2], and applies to an elliptic operator on an arbitrary open set in Euclidean space. We show that our criterion for analyticity is equivalent to the criterion of [3] applied to a compact manifold. Letting
again $A\phi_j = \lambda_j \phi_j$ and $\mu_j = |\lambda_j|$, the criterion of [3] for analyticity of $f = \sum f_j \phi_j$ is: there is a constant $C$ such that for all $k \geq 0$

$$\sum \mu_j^{2k} |f_j|^2 \leq ((km)!)^2 C^{2k+2}. \tag{3}$$

The equivalence of (3) with our main theorem reduces easily to the following:

**Lemma.** Let $0 < \mu_j < \infty$, and suppose $\sum r^{\mu_j} < \infty$ for each $r < 1$. Then the condition (3) on sequences $\{f_j\}$ is equivalent to

$$|f_j| \leq D^{|\mu_j|} \text{ for some } D < \infty \text{ and } t < 1. \tag{4}$$

Note that the condition $\sum r^{\mu_j} < \infty$ has been derived from (1) in the course of our original proof, when $(\mu_j)^m$ is the $j$th eigenvalue of $A$.

To prove the lemma, assume (3). Then each term of the series in the left of (3) is bounded by the right of (3), so $(\mu_j/k)^k |f_j|^{1/m} \leq (mC^{1/m})C^{1/m}$. Stirling’s formula gives, for an appropriate constant $B$, $|f_j|^{1/m}(\mu_j/k)! \leq (B/2)^{k+1}$, so that $|f_j|^{1/m} \sum (\mu_j/k)! \leq B$, i.e. $|f_j| \leq B^m (e^{-1/B})^{\mu_j}$, which is (4).

For the converse, we assume (4) and prove $\sum \mu_j^k |f_j| \leq k! C^{k+1}$, which implies (3). Consider $\psi(z) = \sum e^{z \mu_j}$. Since $\sum r^{\mu_j} < \infty$ for each $r < 1$, $\psi$ is analytic for $\text{Re}(z) < 0$, and thus on the compact set $\{z = \log t\}$ satisfies $|\psi^{(k)}(z)| \leq k! C^{k+1}$, where $\psi^{(k)}$ is the $k$th derivative of $\psi$. Using this, we find $\sum \mu_j^k |f_j| \leq D \sum \mu_j^k |\psi^{(k)}(\log t)| \leq k! C^{k+1}$.

**References**


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