ON THE FRATTINI SUBGROUP

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1. **Introduction.** Let F be a free group, R a normal subgroup of F and S a fully invariant subgroup of R. A question which has attracted some attention recently asks what is the structure of F/S (see for example [**1**], [**2**], [**3**]). In this paper we shall be concerned with the Frattini subgroup of F/S. Recall that the Frattini subgroup $\Phi(G)$ of a group G is the intersection of all the maximal subgroups of G; following Gaschutz [**5**], G will be called Φ-free if $\Phi(G)=1$. Our main result will be

THEOREM 1.1. *If F/R is residually finite*[2] *then F/R' is Φ-free.*

If G/N is Φ-free then $\Phi(G)\leqq N$ [**5**], hence the following corollary is an immediate consequence of Theorem 1.

COROLLARY 1.2. *If F/R is residually finite and S is a fully invariant subgroup of R such that $S\leqq R'$ then $\Phi(F/S)\leqq R'/S$.*

In order to obtain more precise information we must make some assumptions on S. This is done most conveniently in the language of varieties of groups. A variety is a class of groups closed under the operations of taking cartesian products, subgroups and homomorphic images. For a variety $\mathfrak{V}$ and an arbitrary group G we obtain a fully invariant subgroup $V(G)$ of G, namely the intersection of all normal subgroups N of G such that $G/N\in\mathfrak{V}$; it is easily seen that if $\mathfrak{U}$ and $\mathfrak{V}$ are varieties with $\mathfrak{U}\leqq\mathfrak{V}$ then $\mathfrak{U}(G)\geqq\mathfrak{V}(G)$. If X is a free group then $X/V(X)$ is called relatively free in the variety $\mathfrak{V}$. If $\mathfrak{U}$ and $\mathfrak{V}$ are varieties, and we denote by $\mathfrak{U}\mathfrak{V}$ the class of all groups G such that G has a normal subgroup N with $N\in\mathfrak{U}$ and $G/N\in\mathfrak{V}$, then $\mathfrak{U}\mathfrak{V}$ is a variety and $UV(G)=U(V(G))$. We say that a variety is nilpotent if it consists of nilpotent groups and a variety has exponent 0 if it contains $\mathfrak{A}$, the variety of all abelian groups. With these remarks we can now state

THEOREM 1.3. *If F/R is residually finite and $\mathfrak{N}$ is a nilpotent variety of exponent* 0, *then $\Phi(F/N(R))=R'/N(R)$.*

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[2] If $\mathfrak{C}$ is a class of groups, we say that a group is residually $\mathfrak{C}$ if the intersection of all normal subgroups with quotient in $\mathfrak{C}$ is trivial.

Now suppose that $\mathfrak{V}=\mathfrak{V}_1\mathfrak{V}_2 \cdots \mathfrak{V}_n$, with each $\mathfrak{V}_i$ nilpotent. From Theorem 2 of Gilbert Baumslag [2] and the fact that relatively free nilpotent groups are residually finitely generated [8, Theorem 17.41], and that finitely generated nilpotent groups are residually finite [7, Theorem 3], we can deduce that $F/V(F)$ is residually finite. Moreover if $\mathfrak{V}_1$ has exponent 0 and $R=V_2 \cdots V_n(F)$ then since F/R is residually finite and $V(F)=V_1(R)$ we may apply Theorem 1.3 to give

COROLLARY 1.4. *If $\mathfrak{V}=\mathfrak{V}_1\mathfrak{V}_2 \cdots \mathfrak{V}_n$ with $\mathfrak{V}_i$ $(1\leqq i\leqq n)$ nilpotent and $\mathfrak{V}_1$ of exponent* 0 *then $\Phi(F/V(F))=R'/V(F)$. In particular $\Phi(F/V(F))$ is nilpotent.*

A group G is polynilpotent if it has a series $1=N_0<N_1< \cdots <N_r=G$ with each N_i normal in G and N_{i+1}/N_i nilpotent. With $\mathfrak{V}$ as in Corollary 1.4, $F/V(F)$ is polynilpotent. The nilpotency of its Frattini subgroup contrasts strongly with the general situation for finitely generated polynilpotent groups (see P. Hall [6, Theorem 2]).

If $\mathfrak{V}=\mathfrak{A}^n$ for some n then $F/V(F)$ is called a free soluble group and as a special case of Corollary 1.4 we have

COROLLARY 1.5. *Free soluble groups are Φ-free.*

There are many other varieties $\mathfrak{V}$ for which we can determine the Frattini subgroup of $F/V(R)$, using the techniques employed in the proofs of the previous theorems. As a sample and a contrast to Theorem 1.3 we prove

THEOREM 1.6. *Let F/R be residually finite and let $\mathfrak{U}$ be the variety of all center extended by metabelian groups.*[3] *Then $\Phi(F/U(R))=R''/U(R)$.*

2. **The proof of Theorem 1.1.** Let A and B be finite groups and let K be the group of all functions from B to A with coordinatewise multiplication. We define an action of B on K by $f^b(b')=f(b'b^{-1})$ for $f\in K$ and $b, b'\in B$. With this action we can form the splitting extension of K by B; this extension is called the wreath product of A and B and denoted by A wr B. A is called the bottom group, B the top group and K the base group of A wr B.

The Fitting subgroup $F(G)$ of a finite group G is the maximal normal nilpotent subgroup of G. Since the Frattini subgroup of a finite group is nilpotent, we have $\Phi(G)\leqq F(G)$.

We say that H is a Hall subgroup of a finite group G if its order and index are coprime.

[3] That is, all groups in which the second derived group is contained in the center.

LEMMA 2.1. *Let N be a normal abelian Hall subgroup of the finite group G. Then $F(G)$ is contained in the centralizer $C_G(N)$* of *N in G.*

PROOF. $N \leqq F(G)$ by the definition of $F(G)$. Since N is a Hall subgroup of G and $F(G)$ is the direct product of its Sylow subgroups we have $F(G) = N \times Q$ and the result follows.

LEMMA 2.2 (JOHN CAMPBELL [4, THEOREM 8.2 (c)]). *If N is a normal nilpotent Hall subgroup of G then $\Phi(G) = \Phi(N) \times Q$ where* $(|N|, |Q|) = 1$.

Let G be a group, $\mathfrak{D}$ a class of groups and $\Lambda_{\mathfrak{D}}$ a subset of the set of all homomorphisms from G into groups in $\mathfrak{D}$. Then we say that $\Lambda_{\mathfrak{D}}$ discriminates G if for every finite set $g_1, \cdots, g_l$ of nontrivial elements of G there exists a $\phi \in \Lambda_{\mathfrak{D}}$ such that $g_i\phi \neq 1$ $(1 \leqq i \leqq l)$.

We shall assume familiarity with the paper of Gilbert Baumslag [2]. For convenience we restate Theorem 1.1: *If F is a free group, R a normal subgroup of F with F/R residually finite, then* $\Phi(F/R') = 1$.

Let $\mathfrak{C}$ be the set of finite homomorphic images of F/R and put $\mathfrak{D} = \{Z_p \text{ wr } G;\ G \in \mathfrak{C},\ p \text{ a prime and } p \nmid |G|\}$. Then

LEMMA 2.3. *There exists a set $\Lambda_{\mathfrak{D}}$ which discriminates F/R' such that for any $\phi \in \Lambda_{\mathfrak{D}}$ with ϕ: $F/R' \rightarrow Z_p$* wr *G the image $(R/R')\phi$ is contained in the base group of Z_p* wr *G.*

The proof of Lemma 2.3 follows the proof of Theorem 1 of Gilbert Baumslag, using the fact that if A is a free abelian group then the set of all homomorphisms from A into elements of $\{Z_p, p \in \pi$, with π any infinite set of primes$\}$ discriminates A. The details are left for the reader.

For any group G with normal subgroup N we have $\Phi(G/N) \geqq \Phi(G)N/N$; hence it is enough to show that for an arbitrary nontrivial element g in F/R' there exists a homomorphism $\phi \in \Lambda_{\mathfrak{D}}$ such that $g\phi \notin \Phi((F/R')\phi)$. The proof divides into two cases, $g \notin R/R'$ and $g \in R/R'$; we treat both cases simultaneously. Let $g \in F/R'$, $g \notin R/R'$ and let $1 \neq h \in R/R'$. Since the centralizer of R/R' in F/R' is R/R' [1, Theorem 1], there exists an element $f \in R/R'$ such that $[g, f] \neq 1$. By Lemma 2.3 there exists ϕ: $F/R' \rightarrow Z_p$ wr G such that $[g, f]\phi \neq 1 \neq h\phi$. Note that if $H = (F/R')\phi$ and K is the base group of Z_p wr G, then $H \cap K$ is a normal abelian Hall subgroup of H and $f\phi$, $h\phi \in H \cap K$. Since $[g, f]\phi \neq 1$ $g\phi \notin C_H(H \cap K)$ and so by Lemma 2.1 $g\phi$ is not in the Fitting subgroup of H and thus not in $\Phi(H)$. Moreover since $H \cap K$ is an elementary abelian p group, $\Phi(H \cap K) = 1$; hence by Lemma 2.2 $h\phi \notin \Phi(H)$. This completes the proof of the theorem.

3. Proofs of Theorems 1.3 and 1.6. A subset T of a group G is called omissible in G if whenever $gp\{T, S\}=G$ for some subset S of G we have $gp\{S\}=G$. We need the following facts.

3.1. If T is omissible in G then $T \leqq \Phi(G)$.

3.2. If T is a normal subgroup of G and H is a subgroup of G with T omissible in H then T is omissible in G [**4**, Theorem 3.1.1].

3.3. If G is nilpotent then G' is omissible in G [**4**, 3.3.1 Corollary].

PROOF OF THEOREM 1.3. It follows immediately from Corollary 1.2 that $\Phi(F/N(R)) \leqq R'/N(R)$. On the other hand, $R'/N(R)$ is omissible in $R/N(R)$ (3.3) and hence in $F/N(R)$ (3.2) giving that $R'/N(R) \leqq \Phi(F/N(R))$ (3.1).

PROOF OF THEOREM 1.6. Since F/R is residually finite we have F/R' residually finite by [**2**, Theorem 2], and so by Theorem 1.1 F/R'' is Φ-free, that is $\Phi(F/U(R)) \leqq R''/U(R)$. On the other hand, $R'/U(R)$ is nilpotent and as in the proof of Theorem 1.3, $R''/U(R) \leqq \Phi(F/U(R))$.

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