SEMI-ISOMORPHISMS OF CERTAIN INFINITE
PERMUTATION GROUPS

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Let $X$ and $Y$ be infinite cardinal numbers, $S(X)$ the full symmetric
group on a set of cardinal $X$, $A(X)$ the alternating group of finite even
permutations on the same set, and $S(X, Y)$ the subgroup of $S(X)$ of
all permutations moving fewer than $Y$ elements.

A semi-automorphism of a group $G$ is a permutation $T$ of $G$ such
that $(xyx)T = (xT)(yT)(xT)$ for all $x, y \in G$. Semi-isomorphism is
defined similarly. Dinkines [1] and Herstein and Ruchte [2] showed
that any semi-automorphism of $S(X, Y)$ or $A(X)$ was either the re-
striction $T$ of an inner automorphism of $S(X)$, or was of the form
$T(-I)$ where $x(-I) = x^{-1}$ for all $x$. Theorem 11.4.6 of [3] states that
every automorphism of any group $G$ such that $A(X) \subseteq G \subseteq S(X)$ is
the restriction of an inner automorphism of $S(X)$. In the present
paper, we prove the common generalization of these two theorems
whose statement is obvious. (See the corollary at the end.)

**Lemma.** If $Q$ is a subset of $A(X)$ containing all 3-cycles and such that
$x, y \in Q$ imply $xyx \in Q$, then $Q = A(X)$.

**Proof.** If $x \in A(X)$, then $x = c_1c_2 \cdots c_n$, where the $c_i$ are 3-cycles.
If $n = 1$, then $x \in Q$. Induct on $n$. Since

$$x = c_1 \cdots c_n = c_1^{-1}(c_1^{-1}(c_2 \cdots c_n)c_1)c_1$$

and the middle factor is the product of $n - 1$ 3-cycles $c_1^{-1}c_ic_1$, it follows
by induction that $x \in Q$.

**Theorem.** Let $X$ be an infinite cardinal number, $G$ and $H$ subgroups
of $S(X)$ containing $A(X)$, and $T$ a semi-isomorphism of $G$ onto $H$. Then
either

1. $T$ is induced by conjugation by an element of $S(X)$, or
2. $T$ is the product of a mapping of type (1) mapping $G$ onto $H$, and the mapping $-I$ of $H$ onto $H$.

**Proof.** Since the center of $H$ is 1, it follows that $T$ preserves order
and powers [1, Lemma 1]. Let $S(S')$ be the subgroup generated by
all elements of order 2 in $G \langle H \rangle$. Since $A(X)$ is simple, $S$ and $S'$ each
contain $A(X)$. Call elements $x, y \in G$, $S$-conjugate iff $x = y^s$ for some

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1 This brief proof is due to Fletcher Gross.

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$s \in S$. Define $S'$-conjugacy similarly. Then $S$- and $S'$-conjugacy are equivalence relations. Moreover, if $x$ is $S$-conjugate to $y$, then $x = u_n \cdots u_1 y u_1 \cdots u_n$ with $o(u_i) = 2$, so that

$$xT = (u_nT) \cdots (u_1T) (yT) (u_1T) \cdots (u_nT),$$

and $xT$ and $yT$ are $S'$-conjugate. Since the inverse of a semi-isomorphism is also a semi-isomorphism, the converse is also true. Therefore $T$ carries an $S$-conjugate class onto an $S'$-conjugate class.

Let $M$ be the set of 3-cycles. Then

1. $M$ is an $S$-conjugate class of $G$.
2. All elements of $M$ have order 3.
3. $\max o(xyxy) = 5$ for $x, y \in M$.

By earlier remarks, $MT$ satisfies these conditions with $H$ instead of $G$ and $S'$ instead of $S$. If $x = (123)(456) \in MT$, then conjugation by an appropriate $s \in A(X) \subset S'$ gives $y = (132)(478) \in MT$. Then

$$xyx = (123)(46785)$$

contrary to (5). If $x = (123)(456)(789) \cdots \in MT$, then conjugation by some $s \in S'$ yields $y = (132)(457)(689) \cdots \in MT$. But then

$$xyx = (123)(48)(59) \cdots ,$$

contrary to (5). By (4), it follows that all elements of $MT$ are 3-cycles. Hence, by (3), $MT = M$.

Let $Q = \{ x \in A(X) \mid xT \in A(X) \}$. If $x, y \in Q$, then $xyx \in Q$. By the lemma, $Q = A(X)$, that is $A(X)T \subset A(X)$. Using $T^{-1}$ instead of $T$, we have $A(X)T = A(X)$. Thus $T^{-1}A(X)$ is a semi-automorphism. By [1] or [2], $T^{-1}A(X)$ is either an automorphism or an anti-automorphism. But all automorphisms of $A(X)$ are of the form $T_z$ where $T_z$ is conjugation by some element $z \in S(X)$ (see, for example, [3, Theorem 11.4.8]). If $T$ is an anti-automorphism on $A(X)$, then $T(-I)$ is an automorphism, hence $T(-I) = T_z$ and $T = T_z(-I)$ on $A(X)$.

Let $U = TT_z^{-1}$ or $T(-I)T_z^{-1}$ in the above two cases. Then $U$ is a semi-isomorphism of $G$ which is the identity on $A(X)$. The theorem will follow if we can show that $U$ is the identity on $G$.

Suppose that there is an $x \in G$ such that $xU \neq x$. We assert that $x$ can be chosen so that it fixes at least 5 letters. If this is false, then choose $x$ so that it fixes the maximum possible number of letters (at most 4). If $x$ contains an $n$-cycle $n \geq 3$, we can assume that $x = (\cdots 123 \cdots) \cdots$, in which case $(123)x(123)$ fixes all letters fixed by $x$ and the letter 3 in addition. In the other case, $x$ is a product of disjoint 2-cycles, say $x = (12)(34) \cdots$; then $(123)x(123)$ again
fixes 3 and all letters fixed by $x$. Since for $y = (123)$, $(yxy) U = y(xU)y \neq yxy$, we have a contradiction in either case. Hence, as asserted, $x$ can be chosen so that it fixes at least 5 letters.

Now let $x$ be any element of $G$ fixing at least 5 letters. We assert that $xU$ fixes the same letters as $x$. Suppose, in fact, that $x$ fixes 1, 2, 3, 4, and 5, but that $xU$ moves 1. Then, changing notation if necessary, $1(xU) \neq 1, 2, 3, 4$. Now

$$xU = [(12)(34)x(12)(34)]U = (12)(34)(xU)(12)(34),$$

$$xU(12)(34) = (12)(34)(xU).$$

But the left side sends 1 into $1(xU)$, while the right sends it into 2($xU$). This contradiction proves that $xU$ fixes all letters fixed by $x$. Consideration of $U^{-1}$ shows that $xU$ fixes the same letters as $x$.

Let $x$ fix at least 5 letters, and $xU \neq x$. For some letter $i$, $ix = j$, $i(xU) = k$, $k \neq j$. The preceding paragraph implies that $i \neq j$. Let $r \neq i$ or $j$. Now the element $(rij)x(rij)$ fixes $r$ and all letters fixed by $x$, hence at least 5 letters altogether. However $(rij)(xU)(rij) = [(rij)x(rij)]U$ does not fix $r$. This contradicts the preceding paragraph. Thus the theorem is true.

**Corollary.** If $A(X) \subseteq G \subseteq S(X)$ and $T$ is a semi-automorphism of $G$, then there is an element $z \in N_{S(X)}(G)$ such that either $T$ is the automorphism $T_z$ (induced by conjugation by $z$) or the anti-automorphism $T_z(-1)$.

**Bibliography**


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