SOME CRITERIA FOR NILPOTENCY IN GROUPS
AND LIE ALGEBRAS

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We shall say that an automorphism \( \alpha \) is nilpotent or acts nilpotently on a group \( G \) if in the holomorph \( H = [G, \alpha] \) of \( G \) with \( \alpha \), \( \alpha \) is a bounded left Engel element, that is, \( [H, k\alpha] = 1 \) for some natural number \( k \). Here \([H, k\alpha]\) means \([H, (k-1)\alpha]\) with \([H, 0\alpha]\) denoting \( H \).

Let \( G' \) denote the commutator subgroup \([G, G]\), and let \( \Phi(G) \) denote the Frattini subgroup of \( G \). If \( \alpha \) is an automorphism of a nilpotent group \( G \) such that the automorphism \( \alpha \) induced by \( \alpha \) on \( G/G' \) is nilpotent (or with certain restrictions on the exponent of \( G \) on \( G/\Phi(G) \)), then by a well-known theorem of Philip Hall (cf. [6, p. 202]), \( \alpha \) is nilpotent. Here we shall show that the same conclusion follows if we know that the restriction of \( \alpha \) to a suitable subgroup of a nilpotent group is nilpotent. We prove the following two theorems announced in [7].

**Theorem 1.** Let \( G \) be a nilpotent group, let \( \alpha \) be an automorphism of \( G \), let \( F \) be a subgroup of \( G \) stable under \( \alpha \) and such that \( \alpha \) is nilpotent on \( F \). If \( F \) contains its centralizer \( C_G(F) \), then \( \alpha \) is nilpotent on \( G \).

For the statement of Theorem 2 it will be convenient to say that a nilpotent group \( G \) is of height \( k \) if \( k \) is the least nonnegative integer so that for each prime \( p \) and each \( p \)-element \( g \) of \( G \), \( g^p \in G' \).

**Theorem 2.** Let \( G \) be a nilpotent group of height \( k \), let \( \alpha \) be an automorphism of \( G \), let \( F \) be a subgroup of \( G \) stable under \( \alpha \) and such that \( \alpha \) is nilpotent on \( F \). Suppose that \( F \) contains the elements of order 4 of \( C_G(F) \), the elements of order \( p \) of \( C_G(F) \) for all odd primes \( p \), and the torsion-free elements of \( C_G(F) \). Then \( \alpha \) is nilpotent on \( G \).

Theorem 2 includes as a special case a recent result of Blackburn (cf. [1]).

In view of the known results about Engel elements we have the following consequence.

**Corollary 1.** Let \( F \) be a nilpotent normal subgroup of a group \( G \) and suppose that \( G \) is either finite or else solvable with nilpotent Hirsch-Plotkin radical \( H \), and suppose further that either \( F \supseteq C_G(F) \) or that \( F \)
is as in Theorem 2. If for each \( x \in G \), the inner automorphism \( \alpha_x \) determined by \( x \) induces a nilpotent automorphism on \( F \), then \( G \) is nilpotent.

Since Theorem 1 is very closely related to a theorem of Thompson (cf. [3, p. 185]) we include the following generalization of the latter.

**Theorem 3.** A nilpotent group \( G \) of finite height has a characteristic subgroup \( C \) with the following properties:

(i) \( C/Z(C) \) has height at most one provided \( G \) is periodic.
(ii) \( [G, C] \) is contained in the center \( Z(C) \) of \( C \) (and hence \( C \) has class at most two).
(iii) \( C_\delta(C) = Z(C) \).
(iv) Every nonnilpotent automorphism of \( G \) induces a nonnilpotent automorphism of \( C \).

We also develop similar ideas for Lie algebras as follows. We say that a derivation \( \delta \) is nilpotent or acts nilpotently on a Lie algebra \( L \) if in the holomorph \( H = L + \{ \delta \} \) of \( L \) with \( \delta \), \( \delta \) is an Engel element. Then we have for Lie algebras the following analogues to our results for groups:

**Theorem 4.** Let \( F \) be a subalgebra of a nilpotent Lie algebra \( L \) such that \( F \) contains its centralizer \( C_L(F) \). If \( \delta \) is a derivation of \( L \) which maps \( F \) onto \( F \) and is nilpotent on \( F \) then \( \delta \) is nilpotent on \( L \).

**Corollary 2.** Let \( F \) be an ideal of a finite-dimensional Lie algebra \( L \) such that \( F \) contains \( C_L(F) \). If for each \( x \) in \( L \) the inner derivation \( \delta_x \) determined by \( x \) induces a nilpotent derivation on \( F \), then \( L \) is nilpotent.

**Theorem 5.** A nilpotent Lie algebra \( L \) has a characteristic subalgebra \( C \) with the following properties:

(i) \( [L, C] \) is contained in the center \( Z(C) \) of \( C \) and hence \( C \) has class at most two.
(ii) \( C_\delta(C) = Z(C) \).
(iii) Every nonnilpotent derivation of \( L \) induces a nonnilpotent derivation of \( C \).

**Proofs.** We note that easy induction arguments immediately give the following:

1. The holomorph of a nilpotent group with a nilpotent automorphism is nilpotent.
2. A nilpotent automorphism of a nilpotent group of finite exponent (of exponent \( p^k \)) has finite order (of order a power of \( p \)).
3. If \( G \) has a normal series \( G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_n = 1 \) and an automorphism \( \alpha \) so that for \( i = 1, 2, \ldots, n \), \( \alpha \) maps \( G_i \) onto \( G_i \) and
if \( \alpha \) is nilpotent on each factor group \( G_{i-1}/G_i \), then \( \alpha \) is nilpotent on \( G \).

**Proof of Theorem 1.** We consider the case first where \( G' \leq F \). Since \([F, kG] = 1\) and \([F, m\alpha] = 1\) for appropriate natural numbers \( k \) and \( m \), it follows that \( F \) has a normal series \( F = F_0 > F_1 > \cdots > F_n = 1 \) so that if \( H \) denotes the holomorph \([G](\alpha)\) then \([F_i, H] \leq F_{i+1}\) for \( i = 0, 1, \cdots, n - 1 \). Let \( C_i \) denote \( C_0(F_i) \). Then \( C_0 \leq C_1 \leq \cdots \leq C_n = G \) and \( C_0 \leq F \). If \( C_{r+1} \) is the least of the \( C_i \) not in \( F \) we shall show that \( \alpha \) is nilpotent on \( C_{r+1} \) as follows: \([F_r, C_{r+1}] \leq F_r \), so that \([F_r, C_{r+1}], m\alpha\) = 1. Then for \( c \in C_{r+1}, f \in F_r \),

\[
([f, c^{-1}], \alpha)^{\nu}([c, \alpha^{-1}], f)^{\nu}([\alpha, f^{-1}], c)^{\nu} = 1
\]

(cf. (*), p. 201 of [6]), and hence \([f, c^{-1}], \alpha] = ([f, c^{-1}], \alpha)\). It follows that \( [[f, c^{-1}], \alpha], \alpha] = ([f, c^{-1}], \alpha) \) for each \( j > 1 \). Since \([F_r, C_{r+1}], m\alpha] = 1\) it follows that \([C_{r+1}, m\alpha]\) \leq C_r \leq F \) and \([C_{r+1}, 2m\alpha] = 1\). Hence \( \alpha \) is nilpotent on \( C_{r+1} \) and consequently on \( FC_{r+1} \). An induction then gives that \( \alpha \) is nilpotent on \( C_n = G \) and the statement of the theorem is proved in the case where \( G' \leq F \). Now let \( G^2 \) denote \( G' \) and for \( t > 2 \), let \( G^t \) denote \([G^{t-1}, G] \) so that \( G = G^1 \leq G^2 \leq \cdots \). In the general case suppose that \( G' \) is the least member of the lower central series not in \( F \). Then \( \alpha \) is nilpotent on \( FG' \) by what was shown above and an induction gives that \( \alpha \) is nilpotent on \( G \). This proves Theorem 1.

**Proof of Theorem 2.** Let \( G = G_0 > G_1 > \cdots > G_n = 1 \) be an invariant series of \( G \) which includes the members of the lower central series of \( G \) and so that each factor \( G_i/G_{i+1} \) has height at most 1. Let \( r \) be maximal so that \( G_{r+1} \leq F \) and assume inductively that the theorem is true for all \( s < r \). Now \( F \cap C_0(F) \) is central in \( C_0(F) \) and the hypotheses of the theorem hold for \( F \cap C_0(F) \) in \( C_0(F) \). If we can show that \( \alpha \) is nilpotent on \( C_0(F) \), then by (3) \( \alpha \) will be nilpotent on \( FC_0(F) \), and by Theorem 1, \( \alpha \) will be nilpotent on \( G \).

Accordingly we need only consider the case where \( F \) is central in \( G \). Then all the torsion free elements of \( G \) are in \( F \) and we let \( k \) be maximal so that \( F \) contains all the elements of order \( p^k \) of \( G \) for all \( p \) (\( k > 0 \) by hypothesis). Let \( c \) be a \( p \)-element of \( G \), for some \( p \); since \( \alpha \) is nilpotent on \( F \), a suitable \( p \)-th power \( \beta \) of \( \alpha \) is the identity on the Sylow \( p \)-subgroup \( F_p \) of \( F \) by (2). Then \( [c, \beta]^{p^k} = (c^{-1}c^p)^{p^k} \) and since \([c^{-1}, \beta]^{p^k}\) is in \( G_r \), hence in \( F \) and therefore central,

\[
(c^{-1}c^p)^{p^k} = c^{-p}\beta^{p^k}[c^{-1}, \beta^{p^k}]
\]

(where \( C_{p^k, 2} \) is the binomial coefficient), which is \( c^{-p}\beta^{p^k} \) since

\[
[c^{-1}, \beta^{p^k}]^{C_{p^k, 2}} = [c^{-p}, \beta^{p^k}]^{C_{p^k, 1/p}} = 1
\]
(for \( k = 1 \) it is only necessary to consider odd \( p \)). Thus \([c, \beta]\) \( \alpha^k = [c^k, \beta] = 1 \), and hence \([c, \beta] \in F\). Since \( \beta \) is a \( p \)-th power of \( \alpha \) it follows that \( \alpha \) (modulo the centralizer of \( c \) in \( (\alpha) \)) and \( c \) generates a \( p \)-subgroup whose order is bounded in terms of the class of nilpotency and height of \( G \) independent of the element \( c \). Thus \( \alpha \) is nilpotent on \( F_p \cap G_\tau \). Since this is true for each \( p \), \( \alpha \) is nilpotent on \( G_\tau \) and hence by (3) on \( F_{G_\tau} \). By the induction assumption \( \alpha \) is nilpotent on \( G \) and the theorem is proved.

**Proof of Corollary 1.** Suppose first that \( G \) is finite. An induction argument on order gives that all maximal subgroups of \( G \) containing \( F \) are nilpotent. Hence \( G/F \) is solvable and thus \( G \) is solvable. We now consider the case where \( G \) is solvable with nilpotent Hirsch-Plotkin radical \( H \) and consider the subgroups \( F \leq H \leq K \leq G \) where \( K \) is a normal subgroup of \( G \) with \( K/H \) abelian. Then for \( x \in K \), \( \alpha \) is nilpotent on \( H \) by the theorems and hence \( x \) is a left Engel element of \( G \). It follows from Theorem 4 of [4] that \( x \) is in \( H \) and therefore \( K \leq H \); since \( G \) is solvable it follows that \( G \leq H \) and hence that \( G \) is nilpotent, as was to be shown.

**Proof of Theorem 3.** Theorem 1 includes the implication that condition (iii) of Theorem 3 implies condition (iv). Accordingly we need only prove that conditions (i), (ii), and (iii) hold for \( G \). We let \( D \) be a maximal characteristic abelian subgroup of \( G \) and let its centralizer \( C_\alpha(D) \) be denoted by \( H \). We let \( K \) be the complete inverse image of the maximal subgroup of height one of the center of \( G/D \) when \( G \) is periodic, while for \( G \) not periodic \( K \) will be the complete inverse image of the center of \( G/D \) (so that \( [G, K] \leq D \)). We then let \( C \) be \( H \cap K \) and \( Q \) be \( C_\alpha(C) \), noting that \( Q \leq H \) since \( D \leq C \). Since all the above subgroups are characteristic in \( G \), \( D(C \cap Q) \) is characteristic as well as abelian, so that from the maximality of \( D \) it follows that \( C \cap Q \leq D \) and therefore \( H \cap K \cap Q = K \cap Q \leq D \). Furthermore, since \( C \leq C_\alpha(D) \), it follows that \( D \) is in \( C \) and from the maximality of \( D \) that \( D \) is in fact the center of \( C \). Finally the fact that \( K \cap Q \leq D \) implies that \( Q \leq D \); for in the contrary case, modulo \( D \), \( Q \) would be a nontrivial normal subgroup which did not meet the maximal subgroup of height one of the center of \( G \) (modulo \( D \)). Thus conditions (i), (ii), and (iii) are satisfied for \( C \) and the proof of the theorem is complete. It is worthy of notice that in case \( C_\alpha(D) \leq D \), then \( D \) itself satisfies the conditions of the theorem in place of \( C \).

**Proofs of Theorems 4 and 5.** The proof of Theorem 4 is essentially the same as that of Theorem 1 except that the argument to replace the lines following (*) is less complicated, since the Jacobi identity is similar to but simpler than (*). For the proof of Theorem 5 we let \( D \) be a maximal characteristic subalgebra and let \( K \) be the
complete inverse image of the center of $L/D$ and then proceed as in Theorem 3. It should be remarked here that in a similar fashion the arguments on p. 202 of [6] for groups can be recast directly to give the analogous results for Lie algebras (cf. [2]).

**Proof of Corollary 2.** $L$ induces a nilpotent algebra of linear transformations on the vector space $F/F'$ and hence by Engel's theorem (cf. [5] for instance) $L/F$ is nilpotent. Thus $L$ is solvable. By an induction argument $FL'$ is nilpotent and hence by Theorem 4 every $x \in LF'$ is an Engel element of $L$. Then by Engel's theorem again, $L$ is nilpotent.

**References**


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