TOPOLOGICAL HOMOTHETIES ON COMPACT HAUSDORFF SPACES

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1. Introduction and terminology. Let \((X, \rho)\) be a metric space and \(f: X \to X\) such that for some \(c > 0\) there exists a topologically equivalent metric \(\rho^*\) such that \(\rho^*(f(x), f(y)) = c \rho^*(x, y)\) for all \(x, y \in X\). In this case we call \(f\) a topological \(c\)-homothety. If \(X\) is compact metrizable then a simple purely topological condition characterizes this property. In fact in this case we have:

**Theorem 1.1.** Let \(X\) be compact metrizable and \(f: X \to X\) continuous. Then \(f\) is a topological \(c\)-homothety for some \(c \in (0, 1)\) if and only if \(f\) is one-to-one and \(\bigcap f^n [X] = \{a\}\) for some \(a \in X\). (The intersection of all iterated images of \(X\) under \(f\) is a singleton.) The proof of this theorem is in [3].

It is the purpose of this note to extend this result to nonmetrizable compact Hausdorff spaces, replacing the role of metrics by generating families of pseudometrics.

Let \(X\) be a completely regular space. A family \(\mathcal{D} = \{\rho_\alpha | \alpha \in \mathcal{A}\}\) of pseudometrics \(\rho_\alpha(x, y)\) on \(X\) will be called a generating family on \(X\) iff it generates the topology of \(X\). (The system of sets \(B(x, \epsilon, \alpha) = \{y | \rho_\alpha(y, x) < \epsilon\}\) for all \(\alpha \in \mathcal{A}\), \(x \in X\) and \(\epsilon > 0\) forms an open subbase for the topology of \(X\).) The set of all such families will be denoted by \(\mathcal{G}(X)\).

If \(f: X \to X\) is such that for some \(c > 0\) \(\rho_\alpha(f(x), f(y)) = c \rho_\alpha(x, y)\) for all \(\alpha \in \mathcal{A}\) and all \(x, y \in X\), \(f\) is said to be a \(c\)-homothety with respect to \(\mathcal{D} = \{\rho_\alpha | \alpha \in \mathcal{A}\}\) \(\in \mathcal{G}(X)\).

If \(f: X \to X\) is such that for some \(\mathcal{D} \in \mathcal{G}(X)\) \(f\) is a \(c\)-homothety with respect to \(\mathcal{D}\), we say \(f\) is a topological \(c\)-homothety.

Let \(X\) be an abstract set and \(f: X \to X\). We will say \(f\) is a squeezing mapping iff \(\bigcap f^n [X] = \{a\}\) for some \(a \in X\). This property will play a crucial role in our investigations. To illustrate this property we state the following theorem which will be used later:

**Theorem 1.2 (A Converse of Banach’s Contraction Theorem).** Let \(X\) be compact metrizable and \(f: X \to X\) a continuous and squeezing mapping. Then for any \(c \in (0, 1)\) there exists a metric \(\rho(x, y)\)
on $X$, generating the topology of $X$, such that $f$ is a $c$-contraction with respect to $p$.

For the proof, see [4].

Remark only that all spaces considered are Hausdorff we can state our main theorem:

**Theorem.** Let $X$ be compact and $f: X \to X$. Then $f$ is a topological $c$-homothety for some $c \in (0, 1)$ iff $f$ is a homeomorphism into and a squeezing mapping.

As a corollary of this theorem and the theorem proved in [5] we will show that each topological $c$-homothety for $c \in (0, 1)$ can be linearized in some linear topological space as a homothety in the usual sense. Finally we will also show that Theorem 1.2 can be generalized dropping the requirement of metrizability from its hypothesis.

2. **Transformations of families of pseudometrics.** Let $X$ be compact, $0 < c \leq 1$, $\mathcal{D} \in \mathcal{G}(X)$, $f: X \to X$, and $\mathcal{D} = \{\rho_\alpha | \alpha \in \mathbb{A}\}$. For each $\alpha \in \mathbb{A}$ we define

$$\rho_\alpha^*(x, y) = \sup c^n \rho_\alpha(f^n(x), f^n(y)).$$

(The supremum is taken over the set of all nonnegative integers and $f^0(x) = x$.)

The resulting family $\{\rho_\alpha^* | \alpha \in \mathbb{A}\}$ we denote by $\mathcal{D}(f, c)$.

**Lemma 2.1.** Let $X$ be compact, $f: X \to X$ continuous and $\mathcal{D} \in \mathcal{G}(X)$. Then (i) $\mathcal{D}(f, c) \in \mathcal{G}(X)$ for each $c \in (0, 1)$ and $c \rho(f(x), f(y)) \leq \rho(x, y)$ for all $x, y \in X$ and $\rho \in \mathcal{D}(f, c)$,

(ii) if $f$ is also squeezing, then $\mathcal{D}(f, 1) \in \mathcal{G}(X)$.

**Proof.** Let $f: X \to X$ be continuous,

$c \in (0, 1)$ and $\mathcal{D} = \{\rho_\alpha | \alpha \in \mathbb{A}\} \in \mathcal{G}(X)$.

First we have to show that for each $\alpha \in \mathbb{A}$ the function $\rho_\alpha^*(x, y) = \sup c^n \rho_\alpha(f^n(x), f^n(y))$ is a pseudometric. Since $\rho_\alpha$ is bounded there exists, for each $x, y \in X$, $n = n(x, y)$ such that $\rho_\alpha^*(x, y) = c^n \rho_\alpha(f^n(x), f^n(y))$. Now let $x, y, z \in X$ be arbitrary elements. Let $m$ be such that

$$\rho_\alpha^*(x, z) = c^m \rho_\alpha(f^m(x), f^m(z));$$

then we have

$$\rho_\alpha^*(x, y) \geq c^m \rho_\alpha(f^m(x), f^m(y)),$$

$$\rho_\alpha^*(y, z) \geq c^m \rho_\alpha(f^m(y), f^m(z)).$$
Applying the triangle inequality to $p_\alpha$ and the elements $f^m(x), f^m(y), f^m(z)$ we get

$$p_\alpha^*(x, z) = c^{-1} p_\alpha(f^m(x), f^m(z))$$

$$\leq c^{-1} p_\alpha(f^m(x), f^m(y)) + c^{-1} p_\alpha(f^m(y), f^m(z))$$

$$\leq p_\alpha^*(x, y) + p_\alpha^*(y, z),$$

which shows that $p_\alpha^*$ is a pseudometric.

From the definition it follows that

$$p_\alpha^*(f(x), f(y)) = \sup c^{-1} p_\alpha(f^{n+1}(x), f^{n+1}(y))$$

$$= c^{-1} \sup c^{-1} p_\alpha(f^{n+1}(x), f^{n+1}(y))$$

$$\leq c^{-1} \sup c^{-1} p_\alpha(f^m(x), f^m(y)) = c^{-1} p_\alpha^*(x, y)$$

which shows that

$$c p_\alpha^*(f(x), f(y)) \leq p_\alpha^*(x, y).$$

Now we have to show that the original family $\{p_\alpha\}$ and the new family $\{p_\alpha^*\}$ generate the same topology. Because $p_\alpha(x, y) \leq p_\alpha^*(x, y)$ for all $x, y \in X$ and all $\alpha \in \mathbb{A}$, it is only necessary to show that for any converging net $\{x_i\} \to x (i \in D)$ the following proposition is true:

$$\forall \alpha \in \mathbb{A} \forall \epsilon > 0 \exists i \in D [i < j \Rightarrow p_\alpha(x_j, x) < \epsilon]$$

where $\prec$ denotes the partial order in $D$.

If this were not the case, then there would exist $\alpha \in \mathbb{A}$ and $\epsilon > 0$ such that, for each $i \in D$, $p_\alpha^*(x_i, x) \geq \epsilon$ for some $j > i$. This would mean that for each $i \in D$ there would exist a nonnegative integer $n = n(i)$ such that

$$c^n p_\alpha^*(f^m(x_i), f^m(x_j)) \geq \epsilon$$

for some $j > i$.

It follows from this inequality and the fact that $p_\alpha$ is bounded that the function $n(i)$ defined on $D$ must be bounded.

Let $n_1, n_2, \ldots, n_k$ be the values of this function; then considering the system of converging nets

$$f^{n_s}(x_i) \to f^{n_s}(x) \quad \text{for } s = 1, 2, \ldots, k$$

we get the desired contradiction of the above inequality, which proves (i).

To prove (ii) it is only necessary to realize that if $f$ is squeezing then the same contradiction can be obtained even if we put $c = 1$. 
Evidently if
\[ \rho_a(f^{n(i)}(x_j), f^{n(i)}(x)) \geq \epsilon \]
for some \( j \geq i \), then the function \( n(i) \) again must be bounded, otherwise there would exist sequences \( j(k) \) and \( n(k) \) such that
\[ f^{n(k)}(x) \to y \quad \text{and} \quad f^{n(k)}(x_{j(k)}) \to z \quad \text{where} \quad y \neq z \]
which could contradict the fact that \( f^n(X) \) is a singleton.

**Lemma 2.2.** Let \( Z \) be compact, \( X \subseteq Z \), \( D = \{ \rho_\alpha | \alpha \in \mathcal{A} \} \subseteq \mathcal{B}(Z) \), \( f: X \to X \) a homeomorphism of \( X \) into itself such that for some \( c \in (0, 1) \), \( \rho_\alpha(f(x), f(y)) \leq c\rho_\alpha(x, y) \) for all \( x, y \in X \) and all \( \alpha \in \mathcal{A} \), and finally let \( g: Z \to Z \) be continuous and such that \( g(x) = f^{-1}(x) \) for all \( x \in f[X] \).

Then \( f \) is a topological \( c \)-homothety on \( X \).

For the proof we need only verify that \( f \) is a \( c \)-homothety with respect to \( D(g, c) \subseteq \mathcal{B}(Z) \).

3. Factorization of mappings. Let \( X \) be completely regular and \( \mathcal{D} = \{ \rho_\alpha | \alpha \in \mathcal{A} \} \subseteq \mathcal{B}(X) \). With each \( \alpha \in \mathcal{A} \) there is associated an equivalence relation \( R_\alpha \) on \( X \) defined by \( x R_\alpha y \Leftrightarrow \rho_\alpha(x, y) = 0 \). Denoting \( X_\alpha = X/R_\alpha \) and by \( \rho_\alpha(x) \) the class of all elements \( \alpha \)-equivalent to \( x \), we have \( \rho_\alpha: X \to X_\alpha \) for each \( \alpha \in \mathcal{A} \). Each \( X_\alpha \) is canonically endowed with the metric \( \rho_\alpha \) defined by
\[ \rho_\alpha(\rho_\alpha(x), \rho_\alpha(y)) = \rho_\alpha(x, y). \]

Forming the topological product \( \prod_\alpha X_\alpha \) and defining \( i: X \to \prod_\alpha X_\alpha \) by \( i(x) = \{ \rho_\alpha(x) \} \) (a typical element of \( \prod_\alpha X_\alpha \) will be denoted by \( \{ x_\alpha \} \) ), we observe that \( i \) is a continuous injection. If \( X \) is compact then each \( X_\alpha \) is compact and \( i: X \to \prod_\alpha X_\alpha \) is a topological embedding.

If \( f: X \to X \) is such that \( x R_\alpha y \Rightarrow f(x) R_\alpha f(y) \) for all \( \alpha \in \mathcal{A} \) then \( f \) induces a map \( f_\alpha: X_\alpha \to X_\alpha \) on each \( X_\alpha \) defined by
\[ f_\alpha(\rho_\alpha(x)) = \rho_\alpha(f(x)) \quad (\alpha \in \mathcal{A}). \]

Defining \( F: \prod_\alpha X_\alpha \to \prod_\alpha X_\alpha \) by \( F\{ x_\alpha \} = \{ f_\alpha(x_\alpha) \} \) we see that the diagram
\[
\begin{array}{ccc}
X & \xrightarrow{i} & \prod_\alpha X_\alpha \\
\downarrow f & & \downarrow F \\
\prod_\alpha X_\alpha & \xrightarrow{i} & \prod_\alpha X_\alpha
\end{array}
\]
is commutative.
It can be seen easily that if \( f \) is nonexpansive with respect to \( \mathcal{D} \), i.e. \( \rho_a(f(x), f(y)) \leq \rho_a(x, y) \) for all \( x, y \in X \) and all \( \alpha \in \mathbb{A} \), and squeezing, then \( f \) preserves all relations \( R_a \) and the induced mappings \( f_a: X_a \to X_a \) enjoy the same properties on \((X_a, \rho^*_a)\). This result enables us finally to prove our theorem.

**Remark 3.1.** Let \( \{(X_a, \rho_a) \mid \alpha \in \mathbb{A}\} \) be a family of metric spaces and \( Y = \prod_a X_a \) its topological product.

Defining the family \( \{\rho'_a \mid \alpha \in \mathbb{A}\} \) on \( Y \) by \( \rho'_a(\{x_a\}, \{y_a\}) = \rho_a(x_a, y_a) \), we observe easily that this belongs to \( \mathcal{G}(Y) \), and if we replace each metric \( \rho_a \) by a topologically equivalent metric \( \rho_{ia} \) on \( X_a \) then the corresponding family \( \{\rho'^*_a \mid \alpha \in \mathbb{A}\} \) again belongs to \( \mathcal{G}(Y) \).

**4. Proof of the theorem.** If \( X \) is a topological \( c \)-homothety for some \( c \in (0, 1) \) then it follows easily that \( f \) is a homeomorphism and that the images \( f^*\{X\} \) shrink to some point.

If conversely \( f \) is a squeezing homeomorphism, then taking any \( \mathcal{D} \in \mathcal{G}(X) \) and applying Lemma 2.1, we see that \( f \) is nonexpanding with respect to \( \mathcal{D}(f, 1) \in \mathcal{G}(X) \). Therefore our factorization process described above is legitimate and each induced mapping \( f_a: X_a \to X_a \) is nonexpanding and squeezing on \( X_a \). Choosing \( c \in (0, 1) \), Theorem 1.2. yields the existence of a metric \( \rho_a \) on \( X_a \) such that \( f_a \) is a \( c \)-contraction on \( (X_a, \rho_a) \) for each \( (X_a, \rho_a) \) for each \( \alpha \in \mathbb{A} \). Each \( (X_a, \rho_a) \) is compact and can be embedded into the Hilbert cube as a closed subset and, using the result of R. H. Bing [1] which assures that a metric defined on a closed subset of a metrizable space can be extended to the whole space, \( \rho_a \) can be extended over this cube. Denoting by \( (H_a, \rho_a) \) the metric space thus obtained and identifying \( X \) with \( i(X) \subset \prod_a X_a \subset \prod_a H_a = H \) we have the following situation:

\( X \) is a closed subset of \( H = \prod_a H_a \) and \( f: X \to X \) a \( c \)-contraction with respect to the family \( \mathcal{D} = \{\rho_a \mid \alpha \in \mathbb{A}\} \in \mathcal{G}(H) \) (considered here as a family of pseudometrics on \( H \)). Now we observe that \( f(X) \) is closed in \( H \) and since \( H \) is a Tychonov cube, Tietze's extension theorem yields the possibility of extending \( f^{-1}: f(X) \to X \) over \( H \). Denoting such a continuous extension of \( f^{-1} \) by \( g: H \to H \), we have all we need to apply Lemma 2.2. to show that \( f \) is a topological \( c \)-homothety on \( X \).

**5. Linearization of topological homotheties.** Following the ideas of J. de Groot laid down mainly in [2] we will present still another characterization of topological homotheties, using a linear topological space as a space into which the given space will be embedded and where the given mapping will act as a homothety in the usual sense.

**Definition 5.1.** Let \( X \) be completely regular and \( f: X \to X \). We say that \( f \) can be linearized in \( L \) as a \( c \)-homothety iff there exists a linear
topological space $L$, a number $c>0$ and a topological embedding $i: X \to L$ in such a way that $i(f(x)) = ci(x)$ for all $x \in X$.

**Lemma 5.1.** Let $X$ be compact metrizable and $f: X \to X$. Then the following two statements are equivalent:

1. $f$ is a topological $c$-homothety for some $c \in (0, 1)$,
2. $f$ can be linearized in a separable Hilbert space as a $c$-homothety for some $c \in (0, 1)$.

The proof of this lemma is in [5], and we will show that this result can be generalized to the nonmetrizable case.

**Theorem 5.1.** Let $X$ be compact and $f: X \to X$. Then the following two statements are equivalent:

1. $f$ is a topological $c$-homothety on $X$ for some $c \in (0, 1)$,
2. $f$ can be linearized as a $c$-homothety in some linear topological space for some $c \in (0, 1)$.

**Proof.** Suppose (2) is true, then obviously $f$ is continuous and one-to-one and therefore a homeomorphism, and since $c^n i(X)$ shrinks to zero in $L$ as $n \to \infty$, $f$ is also squeezing, thus a topological $c$-homothety.

Suppose now (1) is true, then taking $D = \{\rho_\alpha | \alpha \in A\} \in \mathcal{D}(X)$ with respect to which $f$ is a $c$-homothety, we observe that the corresponding mappings $f_\alpha: X_\alpha \to X_\alpha$ are now $c$-homotheties on the metric spaces $(X_\alpha, \rho_\alpha)$. Using Lemma 5.1, we can consider each $X_\alpha$ to be embedded in a separable Hilbert space $H_\alpha$ in such a way that the mapping $f_\alpha: X_\alpha \to H_\alpha$ acts as multiplying by $c$ in $H_\alpha$. Defining $L = \prod \alpha H_\alpha$ we have established (2). Q.E.D.

Combining this result with our main theorem, our final statement reads as follows:

Let $X$ be compact and $f: X \to X$. Then the following statements are equivalent:

1. $f$ is a topological $c$-homothety for some $c \in (0, 1)$.
2. $f$ is a squeezing homeomorphism.
3. $f$ can be linearized in some linear topological space $L$ as a $c$-homothety for some $c \in (0, 1)$.

**Remark.** It follows from the proof of Theorem 5.1 that statement (3) can be modified to require that $L$ be locally convex.

Now we apply the results of §§2 and 3 to generalize Theorem 1.2.

6. A converse of the generalized Banach's contraction theorem.

**Definition 6.1.** Let $X$ be completely regular and $D \in \mathcal{D}(X)$. If $f: X \to X$ is such that for some $c \in (0, 1)$ we have $\rho(f(x), f(y)) \leq c \rho(x, y)$ for all $\rho \in D$ and all $x, y \in X$ we say that $f$ is a $c$-contraction with respect to $D$. 

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If for \( c \in (0, 1) \) there exists \( D \in \mathcal{G}(X) \) such that \( f \) is a \( c \)-contraction with respect to \( D \) we say that \( f \) is a topological \( c \)-contraction on \( X \).

**Proposition 6.1.** If \( f : X \to X \) is such that for some \( c \in (0, 1) \) there exists \( D \in \mathcal{G}(X) \) such that:

(i) \( f \) is a \( c \)-contraction with respect to \( D \),
(ii) \( X \) is complete with respect to \( D \). (If a sequence is \( \rho \)-Cauchy for all \( \rho \in D \) then it converges.)

Then \( f \) has a unique fixed point \( a \in X \) and \( f^n(x) \to a \) for all \( x \in X \). If we further assume that each \( \rho \in D \) is bounded then \( f \) is squeezing.

The proof of this proposition follows immediately from the fact that for any \( x \in X \) the sequence \( \{f^n(x)\} \) is \( \rho \)-Cauchy for all \( \rho \in D \), and if \( \rho \) is bounded then \( \rho \)-diameters of \( f^n[X] \) tend to zero.

If \( X \) is compact we have as a corollary the following theorem:

**Theorem 6.1.** Let \( X \) be compact and \( f : X \to X \) continuous. Then \( f \) is a topological \( c \)-contraction for any \( c \in (0, 1) \) if and only if \( f \) is squeezing.

**Proof.** If \( f \) is a topological \( c \)-contraction for some \( c \in (0, 1) \) then Proposition 6.1. yields that \( f \) is squeezing. If \( f \) is squeezing and continuous then Lemma 2.1. shows the possibility of choosing a \( D \in \mathcal{G}(X) \) with respect to which \( f \) is nonexpansive. Then the factorization process is legitimate with this \( D \in \mathcal{G}(X) \) and if we denote \( \mathfrak{D} = \{\rho_\alpha \mid \alpha \in \mathfrak{A}\} \) we get that for each \( \alpha \in \mathfrak{A} \) the induced mapping \( f_\alpha : X_\alpha \to X_\alpha \) is continuous and squeezing. Thus choosing \( c \in (0, 1) \) arbitrarily, Theorem 1.2. yields an existence of metrics \( \rho_{1\alpha} \) on each \( X_\alpha \) such that \( f_\alpha \) is a \( c \)-contraction on \( (X_\alpha, \rho_{1\alpha}) \). Identifying \( X \) with \( i(X) \) it follows from Remark 3.1. that there exists a family \( \mathfrak{D}^* \in \mathcal{G}(Y) \) (\( Y = \coprod X_\alpha \)) such that \( F \) is a \( c \)-contraction with respect to it, which yields our theorem.

The author would like to express his gratitude to Professor Michael Edelstein for his valuable advice and improvements.

**References**


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