TOPOLOGICAL HOMOTHETIES ON COMPACT HAUSDORFF SPACES

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1. Introduction and terminology. Let (X, ρ) be a metric space and $f: X \to X$ such that for some $c > 0$ there exists a topologically equivalent metric ρ^* such that $\rho^*(f(x), f(y)) = c\rho^*(x, y)$ for all $x, y \in X$. In this case we call f a *topological c-homothety*. If X is compact metrizable then a simple purely topological condition characterizes this property. In fact in this case we have:

THEOREM 1.1. *Let X be compact metrizable and $f: X \to X$ continuous. Then f is a topological c-homothety for some $c \in (0, 1)$ if and only if f is one-to-one and $\bigcap f^n[X] = \{a\}$ for some $a \in X$. (The intersection of all iterated images of X under f is a singleton.) The proof of this theorem is in* [3].

It is the purpose of this note to extend this result to nonmetrizable compact Hausdorff spaces, replacing the role of metrics by generating families of pseudometrics.

Let X be a completely regular space. A family $\mathfrak{D} = \{\rho_\alpha \mid \alpha \in \mathfrak{A}\}$ of pseudometrics $\rho_\alpha(x, y)$ on X will be called a *generating family* on X iff it generates the topology of X. (The system of sets $B(x, \epsilon, \alpha) = \{y \mid \rho_\alpha(y, x) < \epsilon\}$ for all $\alpha \in \mathfrak{A}$, $x \in X$ and $\epsilon > 0$ forms an open subbase for the topology of X.) The set of all such families will be denoted by $\mathfrak{G}(X)$.

If $f: X \to X$ is such that for some $c > 0$ $\rho_\alpha(f(x), f(y)) = c\rho_\alpha(x, y)$ for all $\alpha \in \mathfrak{A}$ and all $x, y \in X$, f is said to be a *c-homothety with respect to* $\mathfrak{D} = \{\rho_\alpha \mid \alpha \in \mathfrak{A}\} \in \mathfrak{G}(X)$.

If $f: X \to X$ is such that for some $\mathfrak{D} \in \mathfrak{G}(X)$ f is a c-homothety with respect to $\mathfrak{D}$, we say f is a *topological c-homothety*.

Let X be an abstract set and $f: X \to X$. We will say f is a *squeezing* mapping iff $\bigcap f^n[X] = \{a\}$ for some $a \in X$. This property will play a crucial role in our investigations. To illustrate this property we state the following theorem which will be used later:

THEOREM 1.2 (A CONVERSE OF BANACH'S CONTRACTION THEOREM). *Let X be compact metrizable and $f: X \to X$ a continuous and squeezing mapping. Then for any $c \in (0, 1)$ there exists a metric $\rho(x, y)$*

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on X, generating the topology of X, such that f is a c-contraction with respect to ρ.

For the proof, see [**4**].

Remarking only that all spaces considered are Hausdorff we can state our main theorem:

THEOREM. *Let X be compact and $f: X \to X$. Then f is a topological c-homothety for some $c \in (0, 1)$ iff f is a homeomorphism into and a squeezing mapping.*

As a corollary of this theorem and the theorem proved in [**5**] we will show that each topological c-homothety for $c \in (0, 1)$ can be linearized in some linear topological space as a homothety in the usual sense. Finally we will also show that Theorem 1.2 can be generalized dropping the requirement of metrizability from its hypothesis.

2. **Transformations of families of pseudometrics.** Let X be compact, $0 < c \leqq 1$, $\mathfrak{D} \in \mathfrak{G}(X)$, $f: X \to X$, and $\mathfrak{D} = \{\rho_\alpha | \alpha \in \mathfrak{A}\}$. For each $\alpha \in \mathfrak{A}$ we define

$$\overset{*}{\rho}_\alpha(x, y) = \sup c^n \rho_\alpha(f^n(x), f^n(y)).$$

(The supremum is taken over the set of all nonnegative integers and $f^0(x) = x$.)

The resulting family $\{\rho_\alpha^* | \alpha \in \mathfrak{A}\}$ we denote by $\mathfrak{D}(f, c)$.

LEMMA 2.1. *Let X be compact, $f: X \to X$ continuous and $\mathfrak{D} \in \mathfrak{G}(X)$. Then* (i) *$\mathfrak{D}(f, c) \in \mathfrak{G}(X)$ for each $c \in (0, 1)$ and $c\rho(f(x), f(y)) \leqq \rho(x, y)$ for all $x, y \in X$ and $\rho \in \mathfrak{D}(f, c)$,*

(ii) *if f is also squeezing, then $\mathfrak{D}(f, 1) \in \mathfrak{G}(X)$.*

PROOF. Let $f: X \to X$ be continuous,

$$c \in (0, 1) \quad \text{and} \quad \mathfrak{D} = \{\rho_\alpha | \alpha \in \mathfrak{A}\} \in \mathfrak{G}(X).$$

First we have to show that for each $\alpha \in \mathfrak{A}$ the function $\rho_\alpha^*(x, y) = \sup c^n \rho_\alpha(f^n(x), f^n(y))$ is a pseudometric. Since ρ_α is bounded there exists, for each $x, y \in X$, $n = n(x, y)$ such that $\rho_\alpha^*(x, y) = c^n \rho_\alpha(f^n(x), f^n(y))$. Now let $x, y, z \in X$ be arbitrary elements. Let m be such that

$$\overset{*}{\rho}_\alpha(x, z) = c^m \rho_\alpha(f^m(x), f^m(z));$$

then we have

$$\overset{*}{\rho}_\alpha(x, y) \geqq c^m \rho_\alpha(f^m(x), f^m(y)),$$

$$\overset{*}{\rho}_\alpha(y, z) \geqq c^m \rho_\alpha(f^m(y), f^m(z)).$$

Applying the triangle inequality to ρ_α and the elements $f^m(x)$, $f^m(y)$, $f^m(z)$ we get

$$\begin{aligned}\overset{*}{\rho}_\alpha(x, z) &= c^m \rho_\alpha(f^m(x), f^m(z)) \\ &\leqq c^m \rho_\alpha(f^m(x), f^m(y)) + c^m \rho_\alpha(f^m(y), f^m(z)) \\ &\leqq \overset{*}{\rho}_\alpha(x, y) + \overset{*}{\rho}_\alpha(y, z),\end{aligned}$$

which shows that ρ_α^* is a pseudometric.

From the definition it follows that

$$\begin{aligned}\overset{*}{\rho}_\alpha(f(x), f(y)) &= \sup c^n \rho_\alpha(f^{n+1}(x), f^{n+1}(y)) \\ &= c^{-1} \sup c^{n+1} \rho_\alpha(f^{n+1}(x), f^{n+1}(y)) \\ &\leqq c^{-1} \sup c^m \rho_\alpha(f^m(x), f^m(y)) = c^{-1} \overset{*}{\rho}_\alpha(x, y)\end{aligned}$$

which shows that

$$c\overset{*}{\rho}_\alpha(f(x), f(y)) \leqq \overset{*}{\rho}(x, y).$$

Now we have to show that the original family $\{\rho_\alpha\}$ and the new family $\{\rho_\alpha^*\}$ generate the same topology. Because $\rho_\alpha(x, y) \leqq \rho_\alpha^*(x, y)$ for all $x, y \in X$ and all $\alpha \in \mathfrak{A}$, it is only necessary to show that for any converging net $\{x_i\} \to x$ $(i \in D)$ the following proposition is true:

$$\forall \alpha \in \mathfrak{A} \ \forall \epsilon > 0 \ \exists i \in D[i \prec j \Rightarrow \overset{*}{\rho}_\alpha(x_j, x) < \epsilon]$$

where $\prec$ denotes the partial order in D.

If this were not the case, then there would exist $\alpha \in \mathfrak{A}$ and $\epsilon > 0$ such that, for each $i \in D$, $\rho_\alpha^*(x_j, x) \geqq \epsilon$ for some $j \succ i$. This would mean that for each $i \in D$ there would exist a nonnegative integer $n = n(i)$ such that

$$c^{n(i)} \rho_\alpha(f^{n(i)}(x_j), f^{n(i)}(x)) \geqq \epsilon$$

for some $j \succ i$.

It follows from this inequality and the fact that ρ_α is bounded that the function $n(i)$ defined on D must be bounded.

Let $n_1, n_2, \cdots, n_k$ be the values of this function; then considering the system of converging nets

$$f^{n_s}(x_i) \to f^{n_s}(x) \qquad \text{for } s = 1, 2, \cdots, k$$

we get the desired contradiction of the above inequality, which proves (i).

To prove (ii) it is only necessary to realize that if f is squeezing then the same contradiction can be obtained even if we put $c = 1$.

Evidently if

$$\rho_\alpha(f^{n(i)}(x_j), f^{n(i)}(x)) \geqq \epsilon$$

for some $j \succ i$, then the function $n(i)$ again must be bounded, otherwise there would exist sequences $j(k)$ and $n(k)$ such that

$$f^{n(k)}(x) \to y \quad \text{and} \quad f^{n(k)}(x_{j(k)}) \to z \quad \text{where} \quad y \neq z$$

which could contradict the fact that $\bigcap f^n(X)$ is a singleton.

LEMMA 2.2. *Let Z be compact, $X \subset Z$, $\mathfrak{D} = \{\rho_\alpha | \alpha \in \mathfrak{A}\} \in \mathfrak{G}(Z)$, $f: X \to X$ a homeomorphism of X into itself such that for some $c \in (0, 1)$, $\rho_\alpha(f(x), (fy)) \leqq c\rho_\alpha(x, y)$ for all $x, y \in X$ and all $\alpha \in \mathfrak{A}$, and finally let $g: Z \to Z$ be continuous and such that $g(x) = f^{-1}(x)$ for all $x \in f[X]$.*

Then f is a topological c-homothety on X.

For the proof we need only verify that f is a c-homothety with respect to $\mathfrak{D}(g, c) \in \mathfrak{G}(Z)$.

3. **Factorization of mappings.** Let X be completely regular and $\mathfrak{D} = \{\rho_\alpha | \alpha \in \mathfrak{A}\} \in \mathfrak{G}(X)$. With each $\alpha \in \mathfrak{G}$ there is associated an equivalence relation R_α on X defined by $x\ R_\alpha\ y \Leftrightarrow \rho_\alpha(x, y) = 0$. Denoting $X_\alpha = X/R_\alpha$ and by $p_\alpha(x)$ the class of all elements R_α-equivalent to x, we have $p_\alpha: X \to X_\alpha$ for each $\alpha \in \mathfrak{A}$. Each X_σ is canonically endowed with the metric ρ_α' defined by

$$\rho_\alpha'(p_\alpha(x), p_\alpha(y)) = \rho_\alpha(x, y).$$

Forming the topological product $\prod_\alpha X_\alpha$ and defining $i: X \to \prod_\alpha X_\alpha$ by $i(x) = \{p_\alpha(x)\}$ (a typical element of $\prod_\alpha X_\alpha$ will be denoted by $\{x_\alpha\}$), we observe that i is a continuous injection. If X is compact then each X_α is compact and $i: X \to \prod_\alpha X_\alpha$ is a topological embedding.

If $f: X \to X$ is such that $x\ R_\alpha\ y \Rightarrow f(x)\ R_\alpha\ f(y)$ for all $\alpha \in \mathfrak{A}$ then f induces a map $f_\alpha: X_\alpha \to X_\alpha$ on each X_α defined by

$$f_\alpha(p_\alpha(x)) = p_\alpha(f(x)) \qquad (\alpha \in \mathfrak{A}).$$

Defining $F: \prod_\alpha X_\alpha \to \prod_\alpha X_\alpha$ by $F\{x_\alpha\} = \{f_\alpha(x_\alpha)\}$ we see that the diagram

$$\begin{array}{ccc} X & \xrightarrow{i} & \prod_\alpha X_\alpha \\ f\downarrow & & F\downarrow \\ X & \xrightarrow{i} & \prod_\alpha X_\alpha \end{array}$$

is commutative.

It can be seen easily that if f is *nonexpansive* with respect to $\mathfrak{D}$, i.e. $\rho_\alpha(f(x), f(y)) \leqq \rho_\alpha(x, y)$ for all $x, y \in X$ and all $\alpha \in \mathfrak{A}$, and squeezing, then f preserves all relations R_α and the induced mappings $f_\alpha: X_\alpha \to X_\alpha$ enjoy the same properties on $(X_\alpha, \rho_\alpha^*)$. This result enables us finally to prove our theorem.

REMARK 3.1. Let $\{(X_\alpha, \rho_\alpha) \mid \alpha \in \mathfrak{A}\}$ be a family of metric spaces and $Y = \prod_\alpha X_\alpha$ its topological product.

Defining the family $\{\rho_\alpha'' \mid \alpha \in \mathfrak{A}\}$ on Y by $\rho_\alpha''(\{x_\beta\}, \{y_\beta\}) = \rho_\alpha(x_\alpha, y_\alpha)$, we observe easily that this belongs to $\mathfrak{G}(Y)$, and if we replace each metric ρ_α by a topologically equivalent metric $\rho_{1\alpha}$ on X_α then the corresponding family $\{\rho_{1\alpha}'' \mid \alpha \in \mathfrak{A}\}$ again belongs to $\mathfrak{G}(Y)$.

4. **Proof of the theorem.** If X is a topological c-homothety for some $c \in (0, 1)$ then it follows easily that f is a homeomorphism and that the images $f^n[X]$ shrink to some point.

If conversely f is a squeezing homeomorphism, then taking any $\mathfrak{D} \in \mathfrak{G}(X)$ and applying Lemma 2.1, we see that f is *nonexpanding* with respect to $\mathfrak{D}(f, 1) \in \mathfrak{G}(X)$. Therefore our factorization process described above is legitimate and each induced mapping $f_\alpha: X_\alpha \to X_\alpha$ is nonexpanding and squeezing on X_α. Choosing $c \in (0, 1)$, Theorem 1.2. yields the existence of a metric ρ_α on X_α such that f_α is a c-contraction on (X_α, ρ_α) for each (X_α, ρ_α) for each $\alpha \in \mathfrak{A}$. Each (X_α, ρ_α) is compact and can be embedded into the Hilbert cube as a closed subset and, using the result of R. H. Bing [1] which assures that a metric defined on a closed subset of a metrizable space can be extended to the whole space, ρ_α can be extended over this cube. Denoting by (H_α, ρ_α) the metric space thus obtained and identifying X with $i(X) \subset \prod_\alpha X_\alpha \subset \prod_\alpha H_\alpha = H$ we have the following situation:

X is a closed subset of $H = \prod_\alpha H_\alpha$ and $f: X \to X$ a c-contraction with respect to the family $\mathfrak{D} = \{\rho_\alpha \mid \alpha \in \mathfrak{A}\} \in \mathfrak{G}(H)$ (considered here as a family of pseudometrics on H). Now we observe that $f(X)$ is closed in H and since H is a *Tychonov cube*, Tietze's extension theorem yields the possibility of extending $f^{-1}: f(X) \to X$ over H. Denoting such a continuous extension of f^{-1} by $g: H \to H$, we have all we need to apply Lemma 2.2. to show that f is a topological c-homothety on X.

5. **Linearization of topological homotheties.** Following the ideas of J. de Groot laid down mainly in [2] we will present still another characterization of topological homotheties, using a linear topological space as a space into which the given space will be embedded and where the given mapping will act as a homothety in the usual sense.

DEFINITION 5.1. Let X be completely regular and $f: X \to X$. We say that f can be *linearized in* L as a c-homothety iff there exists a linear

topological space L, a number $c>0$ and a topological embedding $i: X \to L$ in such a way that $i(f(x)) = ci(x)$ for all $x \in X$.

LEMMA 5.1. *Let X be compact metrizable and $f: X \to X$. Then the following two statements are equivalent:*

(1) *f is a topological c-homothety for some $c \in (0, 1)$,*

(2) *f can be linearized in a separable Hilbert space as a c-homothety for some $c \in (0, 1)$.*

The proof of this lemma is in [5], and we will show that this result can be generalized to the nonmetrizable case.

THEOREM 5.1. *Let X be compact and $f: X \to X$. Then the following two statements are equivalent:*

(1) *f is a topological c-homothety on X for some $c \in (0, 1)$,*

(2) *f can be linearized as a c-homothety in some linear topological space for some $c \in (0, 1)$.*

PROOF. Suppose (2) is true, then obviously f is continuous and one-to-one and therefore a homeomorphism, and since $c^n i(X)$ shrinks to zero in L as $n \to \infty$, f is also squeezing, thus a topological c-homothety.

Suppose now (1) is true, then taking $\mathfrak{D} = \{\rho_\alpha \mid \alpha \in \mathfrak{A}\} \in \mathfrak{G}(X)$ with respect to which f is a c-homothety, we observe that the corresponding mappings $f_\alpha: X_\alpha \to X_\alpha$ are now c-homotheties on the metric spaces (X_α, ρ_α). Using Lemma 5.1, we can consider each X_α to be embedded in a separable Hilbert space $\mathfrak{H}_\alpha$ in such a way that the mapping $f_\alpha: X_\alpha \to \mathfrak{H}_\alpha$ acts as multiplying by c in $\mathfrak{H}_\alpha$. Defining $L = \prod_\alpha \mathfrak{H}_\alpha$ we have established (2). Q.E.D.

Combining this result with our main theorem, our final statement reads as follows:

Let X be compact and $f: X \to X$. Then the following statements are equivalent:

(1) *f is a topological c-homothety for some $c \in (0, 1)$.*

(2) *f is a squeezing homeomorphism.*

(3) *f can be linearized in some linear topological space L as a c-homothety for some $c \in (0, 1)$.*

REMARK. It follows from the proof of Theorem 5.1 that statement (3) can be modified to require that L be locally convex.

Now we apply the results of §§2 and 3 to generalize Theorem 1.2.

6. A converse of the generalized Banach's contraction theorem.

DEFINITION 6.1. Let X be completely regular and $\mathfrak{D} \in \mathfrak{G}(X)$. If $f: X \to X$ is such that for some $c \in (0, 1)$ we have $\rho(f(x), f(y)) \leq c\rho(x, y)$ for all $\rho \in \mathfrak{D}$ and all $x, y \in X$ we say that f is a *c-contraction* with respect to $\mathfrak{D}$.

If for $c \in (0, 1)$ there exists $\mathfrak{D} \in \mathfrak{G}(X)$ such that f is a c-contraction with respect to $\mathfrak{D}$ we say that f is a *topological c-contraction* on X.

PROPOSITION 6.1. *If $f: X \to X$ is such that for some $c \in (0, 1)$ there exists $\mathfrak{D} \in \mathfrak{G}(X)$ such that:*

(i) *f is a c-contraction with respect to $\mathfrak{D}$,*

(ii) *X is complete with respect to $\mathfrak{D}$. (If a sequence is ρ-Cauchy for all $\rho \in \mathfrak{D}$ then it converges.)*

Then f has a unique fixed point $a \in X$ and $f^n(x) \to a$ for all $x \in X$. If we further assume that each $\rho \in \mathfrak{D}$ is bounded then f is squeezing.

The proof of this proposition follows immediately from the fact that for any $x \in X$ the sequence $\{f^n(x)\}$ is ρ-Cauchy for all $\rho \in \mathfrak{D}$, and if ρ is bounded then ρ-diameters of $f^n[X]$ tend to zero.

If X is compact we have as a corollary the following theorem:

THEOREM 6.1. *Let X be compact and $f: X \to X$ continuous. Then f is a topological c-contraction for any $c \in (0, 1)$ if and only if f is squeezing.*

PROOF. If f is a topological c-contraction for some $c \in (0, 1)$ then Proposition 6.1. yields that f is squeezing. If f is squeezing and continuous then Lemma 2.1. shows the possibility of choosing a $\mathfrak{D} \in \mathfrak{G}(X)$ with respect to which f is nonexpansive. Then the factorization process is legitimate with this $\mathfrak{D} \in \mathfrak{G}(X)$ and if we denote $\mathfrak{D} = \{\rho_\alpha \mid \alpha \in \mathfrak{A}\}$ we get that for each $\alpha \in \mathfrak{A}$ the induced mapping $f_\alpha: X_\alpha \to X_\alpha$ is continuous and squeezing. Thus choosing $c \in (0, 1)$ arbitrarily, Theorem 1.2. yields an existence of metrics $\rho_{1\alpha}$ on each X_α such that f_α is a c-contraction on $(X_\alpha, \rho_{1\alpha})$. Identifying X with $i(X)$ it follows from Remark 3.1. that there exists a family $\mathfrak{D}^* \in \mathfrak{G}(Y)$ $(Y = \prod_\alpha X_\alpha)$ such that F is a c-contraction with respect to it, which yields our theorem.

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