ON ADDITIVE FUNCTIONALS

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Let \( C(S) \) denote the class of bounded real-valued continuous functions defined on a topological space \( S \). In [1] a generalization of the Riesz representation theorem is proven for a class of additive functionals defined on \( C(S) \) where \( S = [0, 1] \). This result was extended to the case where \( S \) is a compact metric space in [2]. In this note we show that the result also holds when \( S \) is a compact Hausdorff space.

Let \( f \in C = C(S) \). We define the norm of \( f \) as \( \|f\| = \max_{s \in S} |f(s)| \). We shall say a functional \( \phi \) defined on \( C \) is an additive functional if \( \phi \) satisfies the following three conditions:

\( (C1) \) Continuity. For each \( \epsilon > 0 \) and \( b > 0 \) there exists \( \delta = \delta(\epsilon, b) > 0 \) such that \( ||f|| \leq b, ||g|| \leq b, \) and \( ||f - g|| \leq \delta \) imply \( |\phi(f) - \phi(g)| \leq \epsilon \).

\( (C2) \) Boundedness. For each \( b > 0 \) there exists \( B = B(b) \) such that \( ||f|| \leq b \) implies \( |\phi(f)| \leq B \).

\( (C3) \) Additivity. If \( f_1 f_2 = 0 \) then \( \phi(f_1 + f_2 + f) = \phi(f_1 + f) + \phi(f_2 + f) - \phi(f) \). (C3) is the generalization of the measure theory equality

\[ m(A_1 \cup A_2 \cup A) = m(A_1 \cup A) + m(A_2 \cup A) - m(A), \]

which holds when \( A_1 \cap A_2 = \emptyset \). In particular, if \( \phi(0) = 0 \), then letting \( f = 0 \) in (C3), we obtain \( \phi(f_1 + f_2) = \phi(f_1) + \phi(f_2) \) when \( f_1 f_2 = 0 \). The representation theorem is stated as follows.

**Theorem.** Let \( S \) be a compact Hausdorff space. Then \( \phi \) is an additive functional on \( C \) if and only if

\[ \phi(f) = \int_S K(f(s), s) \mu(ds), \]

where

(i) \( \mu \) is a measure of finite variation defined on the Borel sets of \( S \),
(ii) \( K(x, s) \) is a measurable function of \( s \) for each \( x \),
(iii) \( K(x, s) \) is a continuous function of \( x \) for \( \mu - a.a.s. \),
(iv) for each \( b > 0 \) there exists \( H = H(b) \) such that \( |x| \leq b \) implies \( |K(x, s)| \leq H \) for \( \mu - a.a.s. \).

The proof of necessity follows as in [1]. The proof of sufficiency is obtained by carrying over the proof in [2] by utilizing the analogy...
between the distance function in a metric space and the uniformity which generates the topology on \( S \). We shall first introduce our notation and then give a brief outline of the proof. Since the complete proof is rather long and is essentially unchanged once the appropriate translation into uniformity notation has been made, we shall only prove the initial lemmas in order to indicate the translation.

Let \( U \) be the uniformity consisting of neighborhoods of the diagonal \( \Delta \) in \( S \times S \). Therefore if \( f \in C \), then \( f \) is uniformly continuous with respect to \( U \). We let \( \eta \) and \( w \) denote open symmetric neighborhoods of \( \Delta \). \( U \) is directed by inclusion and we shall denote limits by \( \lim_{\eta \to \Delta} \).

Given \( a \in S \), let \( \eta(a) = \{ b : (a, b) \in \eta \} \) and \( \eta(A) = \bigcup_{a \in A} \eta(a) \) for \( A \subset S \).

Note that \( \eta(A) \) is always an open set in \( S \). The \( \eta \)-rim about \( A \) is defined as \( R(A, \eta) = A^c \cap \eta(A) \). We write \( g \in R(A, \eta) \) if \( g(x) = 0 \), \( x \in R(A, \eta) \). Thus \( g \in R(A, \eta) \) if \( g \) has its support in \( R(A, \eta) \). Given \( h \) real, we write \( g \in R(A, \eta, h) \) if \( g \in R(A, \eta) \) and \( \|g\| \leq \|h\| \). We let \( P(A, w, h) \) denote the set of \( f \in C \) such that \( \|f\| \leq \|h\| \); there exists \( \eta_1 \) such that \( \eta_1 \circ \eta \subset w \) and \( f(s) = h \) for \( s \in \text{Cl}(\eta_1(A)) \); and \( f(s) = 0 \) for \( s \in w(A)^c \). Briefly, \( f \in P(A, w, h) \) means \( f \) has height \( h \) on a closed neighborhood of \( A \) and \( f \) is zero off of \( w(A) \).

Given disjoint closed sets \( F_1 \) and \( F_2 \), we let \( u \in C \) denote a function such that \( 0 \leq u \leq 1 \), \( u(F_1) = 1 \), and \( u(F_2) = 0 \). The existence of \( u \) is guaranteed by Urysohn's Lemma.

We shall now give an outline of the stages in the proof and shall indicate the lemmas in [2] corresponding to each stage.

It is first shown that \( \lim_{\eta \to \Delta} \phi(f) = \lambda_h(A) \) exists for \( f \in P(A, w, h) \) under the assumption \( \phi(0) = 0 \) (Lemmas 1–3). It is then shown that \( \lambda_h(F), F \in \mathcal{F} \), defines a finitely-additive set function on the closed sets \( \mathcal{F} \) (Lemma 4). Utilizing a standard method, we obtain \( \lambda_h = \lambda_h^+ - \lambda_h^- \) where \( \lambda_h^+ \) and \( \lambda_h^- \) are nonnegative, finitely-additive set functions on \( \mathcal{F} \).

It is then shown that \( \lambda_h^+ \) is a regular content on \( \mathcal{F} \) with an identical proof holding for \( \lambda_h^- \) (Lemmas 5–14). Thus \( \lambda_h^+ \) and \( \lambda_h^- \) can be extended to regular measures \( \mu^+ \) and \( \mu^- \) on the Borel sets \( \mathcal{B} \) of \( S \). Letting \( \mu_h = \mu^+ - \mu^- \), we obtain a regular signed measure on \( \mathcal{B} \) such that \( \mu_h(F) = \lambda_h(F), F \in \mathcal{F} \). We define a measure \( \mu = \sum_{i=1}^\infty (\mu_h^+ + \mu_h^-)/2^i\|\mu_h\| \), where \( \{h_i\} \) is a countable dense set of reals and \( \|\mu_h\| \) is the total variation of \( \mu_h \).

It follows that \( \mu \geq \mu_h \) for each \( h \) (Lemma 16). Thus the Radon-Nikodym Theorem yields a kernel \( K(h, s) \) which is \( \mathcal{B} \)-measurable in \( s \) for each \( h \) such that

\[
\mu_h(B) = \int_s K(h\psi_B(s), s)\mu(ds),
\]

where \( \psi_B \) is the characteristic function of \( B \in \mathcal{B} \). As in the proofs of
Lemmas 10–12 [1], it can now be shown that there exist a measure $\mu_*$ and a kernel $K_*(h, s)$ satisfying (i)–(iv) of the above theorem such that

$$\mu_*(B) = \int_B K_*(h\psi_B(s), s)\mu_*(ds), \quad B \in \mathcal{A}.$$ 

Here $K_*(0, s) = 0$ because we have assumed $\phi(0) = 0$. For $f \in C$ we now define

$$\phi_1(f) = \int_B K_*(f(s), s)\mu_*(ds),$$

and show $\phi(f) = \phi_1(f)$ (Lemmas 17, 18). The assumption that $\phi(0) = 0$ is then easily removed.

We shall now prove the existence of $\lim_{\varepsilon \to 0} \phi(f) = \lambda_*(A)$ where $f \in P(A, h, w)$. The proof of Lemma 1 is immediate.

**Lemma 1.** Let $A \subset S$ and $\eta_1 \circ \eta_1 \subset \eta_2$. If $F_1 = \text{Cl}(\eta_1(A))$ and $F_2 = \eta_2(A)^c$ then $F_1 \cap F_2 = \emptyset$.

**Lemma 2.** Let $f \in C$, $A \subset S$, and $w \in U$. If $g_* \in R(A, w)$ and $\epsilon > 0$ then there exists $g \in C$ and $w \in U$ such that $g$ has support in $R(A, w)^c \cap R(A, w)$ and $\|\phi(f + g) - \phi(f + g_*)\| < \epsilon$.

**Proof.** (C1) implies that if $\|g\| \leq \|g_*\|$ then there exists $\delta > 0$ such that $\|g - g_*\| \leq \delta$ implies $\|\phi(f + g) - \phi(f + g_*)\| \leq \delta$. Choose $\eta_1$ such that $(s, t) \in \eta_1$ implies $\|g_*(s) - g_*(t)\| < \delta/2$. Let $\eta$ satisfy $\eta \circ \eta_1 \subset \eta_1 \cap w$. Set $F_1 = \text{Cl}(\eta_1(A))$ and $F_2 = (\eta_1 \cap w)(A)^c$; hence Lemma 1 implies $F_1 \cap F_2 = \emptyset$. Choose $u \in C$ such that $u(F_1) = 0$ and $u(F_2) = 1$. Let $g = u g_*; \|g\| \leq \|g_*\|$ since $\|u\| = 1$. If $s \in F_1$ then $u(s) = 0$; hence $g$ has support in $R(A, w)^c \cap R(F_1, w)$. If $s \in F_2$ then $g(s) = g_*(s)$. If $s \notin F_2$ then there exists $t \in A$ such that $(s, t) \in w \cap \eta_1$; hence $\|g_*(t) - g_*(s)\| = \|g_*(s)\| \leq \delta/2$. Therefore $\|g(s) - g_*(s)\| = \|u(s)g_*(s) - g_*(s)\| \leq \delta$. Thus $\|g - g_*\| \leq \delta$; hence $\|\phi(f + g) - \phi(f + g_*)\| \leq \epsilon$.

**Lemma 3.** Let $f$ and $A$ be as in Lemma 2. Then

$$\lim_{\varepsilon \to 0} \sup_{w \in \Delta} \|\phi(f + g) - \phi(f)\| = 0.$$ 

**Proof.** Assume false. Therefore there exists $\epsilon > 0$ such that for any $\eta$ there exists $w \subset \eta$ and $g \in R(A, w, h)$ such that $\|\phi(f + g) - \phi(f)\| > \epsilon$.

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2 The referee has pointed out that the proof of Lemma 11 [1] (also Lemma 2: Additive functionals on $L_p$ spaces, Canad. J. Math. 18 (1966), 1264–1271) only proves convergence in measure. However, the method of proof can be modified to yield convergence a.e.
Choose \( w_1 \) and \( g_1^* \in R(A, w_1, h) \) such that \(|\phi(f + g_1^*) - \phi(f)| > \epsilon\). By Lemma 2 there exists \( \eta_1 \) and \( g_1 \) such that \( g_1 \) has support in \( R(A, \eta_1) \cap R(A, w_1) \) and \(|\phi(f + g_1) - \phi(f)| > \epsilon\). Now there exists \( w_2 \subset \eta_1 \) and \( g_2^* \in R(A, w_2, h) \) such that \(|\phi(f + g_2^*) - \phi(f)| > \epsilon\). By Lemma 2 there exists \( \eta_2 \) and \( g_2 \) such that \( g_2 \) has support in \( R(A, \eta_2) \cap R(A, w_2) \) and \(|\phi(f + g_2) - \phi(f)| > \epsilon\). Proceeding inductively, we obtain \( g_i \) with disjoint supports such that \(|\phi(f + g_i) - \phi(f)| > \epsilon\), \( 1 \leq i \leq n \). It is no loss of generality to assume \( \phi(f + g_i) - \phi(f) > \epsilon \) (or \( < - \epsilon \), \( 1 \leq i \leq n \), for otherwise we could choose a subsequence \( g_{i_1}, \ldots, g_{i_n} \). Now (C3) implies

\[
\phi \left( f + \sum_{i=1}^{n} g_i \right) - \phi(f) = \sum_{i=1}^{n} \left[ \phi(f + g_i) - \phi(f) \right] > n \epsilon;
\]

hence \( \phi(f + \sum_{i=1}^{n} g_i) > \phi(f) + n \epsilon \) for all \( n \). Since

\[
|f + \sum_{i=1}^{n} g_i| \leq |f| + |h|,
\]

(C2) is contradicted for \( n \) sufficiently large.

Remark. We shall now assume \( \phi(0) = 0 \). (C3) then implies \( \phi(f_1 + f_2) = \phi(f_1) + \phi(f_2) \) if \( f_1 f_2 = 0 \).

Lemma 4. Let \( f \in P(A, w, h) \). Then \( \lim_{w \to w} \phi(f) = \lambda_A(A) \) exists.

Proof. It suffices to show that for \( \epsilon > 0 \) there exists \( w \) such that \( w_1 \subset w \) and \( w_2 \subset w \) imply \(|\phi(f_1) - \phi(f_2)| < \epsilon\) where \( f_i \in P(A, w_i, h) \), \( i = 1, 2 \). By the above remark and Lemma 3 we can choose \( w' \) such that \( g \in R(A, w', h) \) implies \(|\phi(g)| < \epsilon/6\). Let \( w \) satisfy \( w_1 \subset w \subset w' \). Let \( w_i \) and \( f_i \) be as above, \( i = 1, 2 \). Let \( \eta = \eta_1 \cap \eta_2 \); hence \( f_i(s) = h \) for \( s \in \eta(A) \), \( i = 1, 2 \). Lemma 3 implies there exists \( w_3 \subset w \cap \eta \) such that \( g \in R(A, w_3, h) \) implies \(|\phi(f_i - v) - \phi(f_i)| < \epsilon/3\), \( i = 1, 2 \). Let \( \eta_1 \) satisfy \( \eta_1 \cap \eta_2 \subset w_3 \) and choose \( \eta_1, \eta_2, \) and \( \eta_3 \) such that \( \eta_{i+1} \cap \eta_{i+1} \subset \eta_i \), \( i = 1, 2, 3 \). Let \( F_1 = Cl(\eta_i(A)) \) and \( F_2 = \eta_i(A)^c \). Lemma 1 implies \( F_1 \cap \eta_i(A)^c = \emptyset \). Let \( u_1(F_1) = 1 \) and \( u_1(\eta_i(A)^c) = 0 \). Lemma 1 also implies \( Cl(\eta_i(A)) \cap F_2 = \emptyset \). Let \( u_2(F_2) = 1 \) and \( u_2(Cl(\eta_i(A))) = 0 \). We note that \( u_1 u_2 = 0 \). Since \( \eta_i \cap \eta_3 = 0 \) and \( f_i(s) = h \) for \( s \in \eta(A) \) and \( \eta_1(s) = 0 \) for \( s \in \eta_3(A) \), we have \( u_1 f_i = h u_i, i = 1, 2 \). Let \( y_i = (u_1 + u_2)f_i - h u_i + g_i \), where \( g_i = u_2 f_i, i = 1, 2 \). Since \( u_1 u_2 = 0 \) we have \( \phi(y_i) = \phi(h u_i) + \phi(g_i), i = 1, 2 \); hence \( |\phi(y_i) - \phi(y_2)| \leq |\phi(g_i)| + |\phi(g_2)| \).

We let \( v_i = y_i - y_2 \); hence \( v_i = f_i - v \). We have \( v_i = (1 - u_1 - u_2)f_i \); hence \( u_i(s) \neq 0 \) only if \( s \in F_1 \cap F_2^c \). Now

\[
F_1^c \cap F_2^c = Cl(\eta_1(A))^c \cap \eta_1(A) \subset A^c \cap \eta_1(A) = R(A, \eta_1) \subset R(A, w_3);
\]

hence \(|\phi(y_i) - \phi(f_i)| < \epsilon/3\) by choice of \( w_3 \). Since \( g_i = u_2 f_i \), we have
$g_i(s) = 0$ if $s \in \text{Cl}(\eta_2(A))$. Also $g_i(s) = 0$ if $s \in \omega(A)$. Hence $g_i(s) \neq 0$ only if $s \in \text{Cl}(\eta_2(A)) \cap \omega(A)$. Now

$$\text{Cl}(\eta_2(A)) \cap \omega(A) \subset A^c \cap \omega(A) = R(A, w) \subset R(A, w');$$

hence $|\phi(g_i)| < \epsilon/6$ by choice of $w'$. Therefore we conclude

$$|\phi(f_1) - \phi(f_2)|$$

$$\leq |\phi(f_1) - \phi(y_1)| + |\phi(y_1) - \phi(y_2)| + |\phi(y_2) - \phi(f_2)|$$

$$< \epsilon/3 + |\phi(g_1)| + |\phi(g_2)| + \epsilon/3 < \epsilon.$$

We have now verified the existence of $\lambda_h(A), A \subset S$, for each $h$. The remainder of the proof follows [2] as outlined above with appropriate translation into uniformity notation.

**References**
