ARCS IN INVERSE LIMITS ON $[0, 1]$ WITH ONLY ONE BONDING MAP

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1. Introduction. In this note, by a continuum we mean a non-degenerate, compact, connected metric space. It is known ([2] or [5]) that a continuum is chainable if and only if it is homeomorphic to the inverse limit of a sequence of maps from $[0, 1]$ onto $[0, 1]$. Indeed, the mappings may be required to be piecewise linear. Henderson has shown that a pseudo arc can be obtained as an inverse limit on $[0, 1]$ with only one bonding map [3], and the author has shown that not every chainable continuum can be so obtained [4]. We now show that if $M$ is an inverse limit on $[0, 1]$ with only one bonding map, and the bonding map is piecewise monotone, then every subcontinuum of $M$ contains an arc. Thus the set of all such continua is a proper subset of the set of chainable continua which are inverse limits on $[0, 1]$ with only one bonding map.

2. Definitions and notation. If each term of the sequence $g = \{g_i\}$ maps $[0, 1]$ onto $[0, 1]$, then the inverse limit of $g$, denoted by $\lim g$, is the subspace of the infinite cartesian product $[0, 1]^\omega$ consisting of all number sequences $\{x_i\}$ such that for each $i > 0$, $g_i(x_{i+1}) = x_i$. If $f$ maps $[0, 1]$ onto $[0, 1]$, then $\lim f$ denotes $\lim g$ where $g = f, f, \ldots$. By an interval we mean a nondegenerate closed subinterval of $[0, 1]$, and the statement that $A$ is an inverse sequence (for $f$) means that $A$ is a sequence $A_1, A_2, \ldots$ such that if $i > 0$, $A_i$ is degenerate or an interval and $f(A_{i+1}) = A_i$. The set of all points of $\lim f$ such that for each $i > 0$, $x_i \subseteq A_i$ is denoted by $\lim(f, A)$. By a subinverse sequence of $A$ is meant an inverse sequence $B = \{B_i\}$ such that if $i > 0$, $B_i \subseteq A_i$. If each of $I$ and $I'$ is an interval, the statement that $f$ maps $I$ onto $I'$ efficiently means that $f(I) = I'$ and no interior point of $I$ maps onto an endpoint of $I'$.

3. Arcs in chainable continua. Our object in this section is to prove the following:

**Theorem.** If $f$ is a piecewise monotone function from $[0, 1]$ onto $[0, 1]$ then each subcontinuum of $\lim f$ contains an arc.

**Proof.** Suppose there is a subcontinuum of $\lim f$ which contains no arc. Then there is an inverse sequence $A = \{A_i\}$ such that $\lim(f, A)$ contains no arc. There is an increasing number sequence $x_0 = 0, x_1$, 

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\[ \cdots, x_n = 1 \] such that if \( 0 < i \leq n, f \big| _{[x_{i-1}, x_i]} \) is monotone. Let \( M \) denote the set of images under \( f \) of the numbers in this sequence and let \( \varepsilon \) denote a positive number such that if each of \( I \) and \( I' \) is an interval, and \( f \) maps \( I \) onto \( I' \) efficiently, and the length of \( I' \) is less than \( \varepsilon \), then \( f \) is monotone on \( I \). Let \( K \) denote the subset of \([0, 1]\) to which a number \( x \) belongs if and only if there is a point \( \{p_i\} \) in \( \lim f \) such that \( p_1 = x \) and such that either (1) for some \( i > 0, p_i \in M \) or (2) for some \( i > 0, f^{-1}(p_i) \) contains an interval. \( K \) is countable and so there is a finite collection \( G' \) of nonoverlapping intervals filling up \([0, 1]\) such that each interval in \( G' \) is of length less than \( \varepsilon/2 \), and no endpoint of an interval in \( G' \), other than 0 or 1, is a number in \( K \). Let \( G \) denote the intervals in \( G' \) which have neither 0 nor 1 as an endpoint. We now state an easily established lemma.

**Lemma 1.** If \( g \) maps \([0, 1]\) onto \([0, 1]\) and \( \alpha = \{\alpha_i\} \) is an inverse sequence for \( g \), and \( I \) is a subinterval of a term \( \alpha_i \) of \( \alpha \), then there is a subinverse sequence \( \{\beta_i\} \) of \( \alpha \) such that \( \beta_i = I \) and for each \( i > n, g \) maps \( \beta_{i+1} \) onto \( \beta_i \) efficiently.

Applying Lemma 1, there is a subinverse sequence \( A^0 = \{A^0(i)\} \) of \( A \) and an integer \( n_0 \) such that for each \( i \geq n_0, f \) maps \( A^0(i+1) \) efficiently onto \( A^0(i) \). If there is an \( n \) such that for \( i > n, f \) is monotone on \( A^0(i) \), then \( \lim(f, A^0) \) is an arc (see [1]). Since \( \lim(f, A^0) \) is not an arc, there is a subsequence of \( A^0 \) each term of which is of length at least \( \varepsilon \) and so there is an interval \( g_1 \) in \( G \) which is a subset of each term of some subsequence of \( A^0 \). Applying Lemma 1 again, there is a subinverse sequence \( A^1 = \{A^1(i)\} \) of \( A^0 \) and an integer \( n_1 \) such that \( A^1(n_1) = g_1 \) and such that if \( i \geq n_1, f \) maps \( A^1(i+1) \) efficiently onto \( A^1(i) \). As before we note that since \( \lim(f, A^1) \) is not an arc, there is an interval \( g_2 \) in \( G \) which is a subset of each term of some subsequence of \( A^1 \). We continue to establish the existence of a sequence \( A^0, A^1, A^2, \ldots \), an increasing integer sequence \( n_1, n_2, \ldots \) and a sequence \( g_1, g_2, \ldots \) of members of \( G \) such that for each \( i > 0, (1) A^i \) is a subinverse sequence of \( A^{i-1} \), (2) \( A^i(n_i) = g_i \), (3) if \( j \geq n_i, f \) maps \( A^i(j+1) \) efficiently onto \( A^i(j) \) and (4) \( g_i \) is a subset of each term of a subsequence of \( A^{i-1} \). There is an integer \( n \) and an integer \( m < n \) such that \( g_n = g_m \). The sequence \( A^m \) is a subinverse sequence of \( A^0 \), \( A^m(n_m) = g_m \), and \( g_m \) is a subset of each term of a subsequence of \( A^m \). Further, as a consequence of the definition of \( G \) we have that if \( j \geq n_m, f \) is not constant on any interval which contains an endpoint of \( A^m(j+1) \) and it follows that \( A^m(j+1) \) is a component of \( f^{-1}[A^m(j)] \). Let \( N \) denote an integer such that \( A^m(n_m+N) \) contains \( A^m(n_m) \). We shall show that if \( j > 0, A^m(n_m+N+j) \) contains \( A^m(n_m+j) \). Either \( A^m(n_m+N+1) \)
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Let \( f^{-1}[A^m(n_m + N)] \) and \( A^m(n_m + N + 1) \) be a component of \( f^{-1}[A^m(n_m + N)] \). Let \( k \) denote a positive integer such that \( A^m(n_m + N + k) \) contains \( g_m \). Since \( f^N(g_m) \subseteq g_m \), then \( f^N[A^m(n_m + N + k)] \) intersects \( g_m \) and thus intersects \( A^m(n_m + N + k) \). But

\[
\begin{align*}
\Rightarrow A^m(n_m + N + k) \quad \text{intersects} \quad A^m(n_m + k),
\end{align*}
\]

so \( A^m(n_m + N + k) \) intersects \( A^m(n_m + k) \) which implies that \( A^m(n_m + N + 1) \) intersects, and thus contains, \( A^m(n_m + 1) \). It follows inductively that if \( j > 0 \), \( A^m(n_m + N + j) \) contains \( A^m(n_m + j) \). Now let \( g = f^N \) and for each \( j > 0 \) let \( B(j) = A^m(n_m + (j - 1)N) \). Then \( g \) is a piecewise monotone map of \([0, 1]\) onto \([0, 1]\), \( B = \{ B(i) \} \) is an inverse sequence (for \( g \)), and for each \( j > 0 \), \( B(j + 1) \supseteq B(j) \) and \( g \) maps \( B(j + 1) \) efficiently onto \( B(j) \). Further, \( \lim(g, B) \) is homeomorphic to a subcontinuum of \( \lim(f, A) \) and so the following lemma applies to complete our proof.

**Lemma 2.** If \( g \) maps \([0, 1]\) onto \([0, 1]\) and is piecewise monotone and \( B = \{ B_i \} \) is an inverse sequence (for \( g \)) such that for each \( i > 0 \), \( B_i \) is a subinterval of \( B_{i+1} \) and \( g \) maps \( B_{i+1} \) efficiently onto \( B_i \), then \( \lim(g, B) \) contains an arc.

**Proof.** For each \( i > 0 \), let \( B_i = [a_i, b_i] \). The sequence \( \{a_i\} \) is non-decreasing and converges to a number \( a \). Similarly, \( \{b_i\} \) converges to a number \( b \). Further, there is a number \( x \) such that the point \((x, x, \ldots)\) is in \( \lim(g, B) \). Suppose that there is a positive integer \( n \) such that \( g(b_{n+2}) = a_{n+1} \) and \( g(a_{n+1}) = a_n \). According as \( g(b_{n+2}) = b_{n+2} \) or \( g(b_{n+2}) = a_{n+2} \), we have that either \( g([b_{n+2}, b_{n+3}]) \) contains \( x \) and is nondegenerate or \( g([a_{n+4}, a_{n+1}]) \) contains \( x \) and is nondegenerate. But \( g \) is piecewise monotone and there do not exist infinitely many mutually disjoint intervals such that if \( I \) is one of them, \( g(I) \) contains \( x \) and is nondegenerate. So there is an integer \( N' \) such that if \( n > N' \), then it is not true that \( g(b_{n+2}) = a_{n+1} \) and \( g(a_{n+1}) = a_n \). It follows that there is an integer \( N \) such that either (1) if \( n > N \), \( g(a_{n+1}) = a_n \), or (2) if \( n > N \), \( g(a_{n+1}) = b_n \). Suppose first that (1) holds and that \( b \neq x \). There is a number \( u, x < u < b \), such that \( g \) is monotone on \([u, b] \) and an integer \( k \) such that if \( j \geq k \), \( b_j > u \). If \( g(u) \leq u \), there is a point \( \{u_i\} \) in \( \lim g \) such that \( u_i = u \) and for \( i > 0 \), \( u_i \leq u_{i+1} < b_{i+1} \). Then for \( i > 0 \), \( g([u_{i+1}, b_{k+1}]) = [u_i, b_{k+1}] \) and \( g \) is monotonic on \([u_i, b_{k+1}] \). It follows that the set of all points \( \{p_i\} \) of \( \lim g \) such that for \( i > 0 \), \( p_{k+1} \in [u_i, b_{k+1}] \) is an arc which is a subset of \( \lim(g, B) \). If \( g(u) > u \), then there is a number \( s \) and a number \( t \) such that (1) \( g(s) = s \), (2) \( x \leq s < t < u \), (3) if \( y \in [s, t] \), \( g(y) \geq y \) and (4) \( g \) is monotonic on \([s, t] \). There
is a point \( \{t_i\} \) in \( \text{lim } g \) such that \( t_1 = t \) and for each \( i > 0, s < t_{i+1} \leq t_i \). For each \( i > 0, g([s, t_{i+1}]) = [s, t_i] \) and \( g \) is monotonic on \([s, t_{i+1}]\), and it follows that the set of all points \( \{\tilde{p}_i\} \) of \( \text{lim } g \) such that for \( i \geq 0, \tilde{p}_{k+i} \in [s, t_{i+1}] \subseteq [a_{k+i}, b_{k+i}] \) is an arc which is a subset of \( \text{lim}(g, B) \). Thus we have that \( \text{lim}(g, B) \) contains an arc in case there is an integer \( N \) such that if \( n > N, g(a_{n+1}) = a_n \) and \( b \neq x \). An analogous argument suffices in case \( a \neq x \) and it remains to show that \( \text{lim}(g, B) \) contains an arc in case there is an integer \( N \) such that if \( n > N, g(a_{n+1}) = b_n \). In this case we consider the piecewise monotone map \( g^2 \) of \([0, 1] \) onto \([0, 1] \) and the inverse sequence \( B' = B_1, B_2, B_3, \ldots \). There is an integer \( k \) such that if \( i > k, g^2(a_{2i+1}) = a_{2i-1} \), so an argument analogous to that given above applies to show that \( \text{lim}(g^2, B') \) contains an arc. But \( \text{lim}(g, B) \) is homeomorphic to \( \text{lim}(g^2, B') \) and this completes our proof.

References


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