ARCS IN INVERSE LIMITS ON [0, 1] WITH ONLY ONE BONDING MAP

WILLIAM S. MAHAVIER

1. Introduction. In this note, by a continuum we mean a nondegenerate, compact, connected metric space. It is known ([**2**] or [**5**]) that a continuum is chainable if and only if it is homeomorphic to the inverse limit of a sequence of maps from [0, 1] onto [0, 1]. Indeed, the mappings may be required to be piecewise linear. Henderson has shown that a pseudo arc can be obtained as an inverse limit on [0, 1] with only one bonding map [**3**], and the author has shown that not every chainable continuum can be so obtained [**4**]. We now show that if M is an inverse limit on [0, 1] with only one bonding map, and the bonding map is piecewise monotone, then every subcontinuum of M contains an arc. Thus the set of all such continua is a proper subset of the set of chainable continua which are inverse limits on [0, 1] with only one bonding map.

2. **Definitions and notation.** If each term of the sequence $g=\{g_i\}$ maps [0, 1] onto [0, 1], then the inverse limit of g, denoted by lim g, is the subspace of the infinite cartesian product $[0, 1]^\infty$ consisting of all number sequences $\{x_i\}$ such that for each $i>0$, $g_i(x_{i+1})=x_i$. If f maps [0, 1] onto [0, 1], then lim f denotes lim g where $g=f, f, \cdots$. By an interval we mean a nondegenerate closed subinterval of [0, 1], and the statement that A is an inverse sequence (for f) means that A is a sequence $A_1, A_2, \cdots$ such that if $i>0$, A_i is degenerate or an interval and $f(A_{i+1})=A_i$. The set of all points of lim f such that for each $i>0$, $x_i\in A_i$ is denoted by lim(f, A). By a subinverse sequence of A is meant an inverse sequence $B=\{B_i\}$ such that if $i>0$, $B_i\subseteq A_i$. If each of I and I' is an interval, the statement that f maps I onto I' efficiently means that $f(I)=I'$ and no interior point of I maps onto an endpoint of I'.

3. **Arcs in chainable continua.** Our object in this section is to prove the following:

THEOREM. *If f is a piecewise monotone function from* [0, 1] *onto* [0, 1] *then each subcontinuum of* lim f *contains an arc.*

PROOF. Suppose there is a subcontinuum of lim f which contains no arc. Then there is an inverse sequence $A=\{A_i\}$ such that lim(f, A) contains no arc. There is an increasing number sequence $x_0=0$, x_1,

Received by the editors June 3, 1968.

$\cdots, x_n=1$ such that if $0<i\leqq n$, $f|[x_{i-1}, x_i]$ is monotone. Let M denote the set of images under f of the numbers in this sequence and let ϵ denote a positive number such that if each of I and I' is an interval, and f maps I onto I' efficiently, and the length of I' is less than ϵ, then f is monotone on I. Let K denote the subset of $[0, 1]$ to which a number x belongs if and only if there is a point $\{p_i\}$ in $\lim f$ such that $p_1=x$ and such that either (1) for some $i>0$, $p_i\in M$ or (2) for some $i>0$, $f^{-1}(p_i)$ contains an interval. K is countable and so there is a finite collection G' of nonoverlapping intervals filling up $[0, 1]$ such that each interval in G' is of length less than $\epsilon/2$, and no endpoint of an interval in G', other than 0 or 1, is a number in K. Let G denote the intervals in G' which have neither 0 nor 1 as an endpoint. We now state an easily established lemma.

LEMMA 1. *If g maps $[0, 1]$ onto $[0, 1]$ and $\alpha=\{\alpha_i\}$ is an inverse sequence for g, and I is a subinterval of a term α_n of α, then there is a subinverse sequence $\{\beta_i\}$ of α such that $\beta_n=I$ and for each $i>n$, g maps β_{i+1} onto β_i efficiently.*

Applying Lemma 1, there is a subinverse sequence $A^0=\{A^0(i)\}$ of A and an integer n_0 such that for each $i\geqq n_0$, f maps $A^0(i+1)$ efficiently onto $A^0(i)$. If there is an n such that for $i>n$, f is monotone on $A^0(i)$, then $\lim(f, A^0)$ is an arc (see [1]). Since $\lim(f, A^0)$ is not an arc, there is a subsequence of A^0 each term of which is of length at least ϵ and so there is an interval g_1 in G which is a subset of each term of some subsequence of A^0. Applying Lemma 1 again, there is a subinverse sequence $A^1=\{A^1(i)\}$ of A^0 and an integer n_1 such that $A^1(n_1)=g_1$ and such that if $i\geqq n_1$, f maps $A^1(i+1)$ efficiently onto $A^1(i)$. As before we note that since $\lim(f, A^1)$ is not an arc, there is an interval g_2 in G which is a subset of each term of some subsequence of A^1. We continue to establish the existence of a sequence $A^0, A^1, A^2, \cdots$, an increasing integer sequence $n_1, n_2, \cdots$ and a sequence $g_1, g_2, \cdots$ of members of G such that for each $i>0$, (1) A^i is a subinverse sequence of A^{i-1}, (2) $A^i(n_i)=g_i$, (3) if $j\geqq n_i$, f maps $A^i(j+1)$ efficiently onto $A^i(j)$ and (4) g_i is a subset of each term of a subsequence of A^{i-1}. There is an integer u and an integer $m<u$ such that $g_u=g_m$. The sequence A^m is a subinverse sequence of A^0, $A^m(n_m)=g_m$, and g_m is a subset of each term of a subsequence of A^m. Further, as a consequence of the definition of G we have that if $j\geqq n_m$, f is not constant on any interval which contains an endpoint of $A^m(j+1)$ and it follows that $A^m(j+1)$ is a component of $f^{-1}[A^m(j)]$. Let N denote an integer such that $A^m(n_m+N)$ contains $A^m(n_m)$. We shall show that if $j>0$, $A^m(n_m+N+j)$ contains $A^m(n_m+j)$. Either $A^m(n_m+N+1)$

$\supseteq A^m(n_m+1)$ or they do not intersect since $A^m(n_m+1)$ is a connected subset of $f^{-1}[A^m(n_m+N)]$ and $A^m(n_m+N+1)$ is a component of $f^{-1}[A^m(n_m+N)]$. Let k denote a positive integer such that $A^m(n_m+N+k)$ contains g_m. Since $f^N(g_m)\subseteq g_m$, then $f^N[A^m(n_m+N+k)]$ intersects g_m and thus intersects $A^m(n_m+N+k)$. But

$$f^N[A^m(n_m + N + k)] = A^m(n_m + k),$$

so $A^m(n_m+N+k)$ intersects $A^m(n_m+k)$ which implies that $A^m(n_m+N+1)$ intersects, and thus contains, $A^m(n_m+1)$. It follows inductively that if $j>0$, $A^m(n_m+N+j)$ contains $A^m(n_m+j)$. Now let $g=f^N$ and for each $j>0$ let $B(j)=A^m[n_m+(j-1)N]$. Then g is a piecewise monotone map of $[0, 1]$ onto $[0, 1]$, $B=\{B(i)\}$ is an inverse sequence (for g), and for each $j>0$, $B(j+1)\supseteq B(j)$ and g maps $B(j+1)$ efficiently onto $B(j)$. Further, $\lim(g, B)$ is homeomorphic to a subcontinuum of $\lim(f, A)$ and so the following lemma applies to complete our proof.

Lemma 2. *If g maps $[0, 1]$ onto $[0, 1]$ and is piecewise monotone and $B=\{B_i\}$ is an inverse sequence (for g) such that for each $i>0$, B_i is a subinterval of B_{i+1} and g maps B_{i+1} efficiently onto B_i, then $\lim(g, B)$ contains an arc.*

Proof. For each $i>0$, let $B_i=[a_i, b_i]$. The sequence $\{a_i\}$ is nonincreasing and converges to a number a. Similarly, $\{b_i\}$ converges to a number b. Further, there is a number x such that the point $(x, x, \cdots)$ is in $\lim(g, B)$. Suppose that there is a positive integer n such that $g(b_{n+2})=a_{n+1}$ and $g(a_{n+1})=a_n$. According as $g(b_{n+3})=b_{n+2}$ or $g(b_{n+3})=a_{n+2}$, we have that either $g([b_{n+2}, b_{n+3}])$ contains x and is nondegenerate or $g([a_{n+3}, a_{n+1}])$ contains x and is nondegenerate. But g is piecewise monotone and there do not exist infinitely many mutually disjoint intervals such that if I is one of them, $g(I)$ contains x and is nondegenerate. So there is an integer N' such that if $n>N'$, then it is not true that $g(b_{n+2})=a_{n+1}$ and $g(a_{n+1})=a_n$. It follows that there is an integer N such that either (1) if $n>N$, $g(a_{n+1})=a_n$, or (2) if $n>N$, $g(a_{n+1})=b_n$. Suppose first that (1) holds and that $b\neq x$. There is a number u, $x<u<b$, such that g is monotone on $[u, b]$ and an integer k such that if $j\geqq k$, $b_j>u$. If $g(u)\leqq u$, there is a point $\{u_i\}$ in $\lim g$ such that $u_1=u$ and for $i>0$, $u_i\leqq u_{i+1}<b_{k+i}$. Then for $i>0$, $g([u_{i+1}, b_{k+i}])=[u_i, b_{k+i-1}]$ and g is monotonic on $[u_{i+1}, b_{k+i}]$. It follows that the set of all points $\{p_i\}$ of $\lim g$ such that for $i>0$, $p_{k+i}\in[u_{i+1}, b_{k+i}]$ is an arc which is a subset of $\lim(g, B)$. If $g(u)>u$, then there is a number s and a number t such that (1) $g(s)=s$, (2) $x\leqq s<t<u$, (3) if $y\in[s, t]$, $g(y)\geqq y$ and (4) g is monotonic on $[s, t]$. There

is a point $\{t_i\}$ in lim g such that $t_1 = t$ and for each $i > 0$, $s < t_{i+1} \leqq t_i$. For each $i > 0$, $g([s, t_{i+1}]) = [s, t_i]$ and g is monotonic on $[s, t_{i+1}]$, and it follows that the set of all points $\{p_i\}$ of lim g such that for $i \geqq 0$, $p_{k+i} \in [s, t_{i+1}] \subseteq [a_{k+i}, b_{k+i}]$ is an arc which is a subset of $\lim(g, B)$. Thus we have that $\lim(g, B)$ contains an arc in case there is an integer N such that if $n > N$, $g(a_{n+1}) = a_n$ and $b \neq x$. An analogous argument suffices in case $a \neq x$ and it remains to show that $\lim(g, B)$ contains an arc in case there is an integer N such that if $n > N$, $g(a_{n+1}) = b_n$. In this case we consider the piecewise monotone map g^2 of $[0, 1]$ onto $[0, 1]$ and the inverse sequence $B' = B_1, B_3, B_5, \cdots$. There is an integer k such that if $i > k$, $g^2(a_{2i+1}) = a_{2i-1}$, so an argument analogous to that given above applies to show that $\lim(g^2, B')$ contains an arc. But $\lim(g, B)$ is homeomorphic to $\lim(g^2, B')$ and this completes our proof.

References

1. C. E. Capel, *Inverse limit spaces*, Duke Math. J. **21** (1954), 233–245.

2. H. Freudenthal, *Entwicklungen von Räumen und ihren Gruppen*, Compositio Math. **4** (1937), 145–234.

3. G. W. Henderson, *The pseudo-arc as an inverse limit with one binding map*, Duke Math. J. **31** (1964), 421–425.

4. W. S. Mahavier, *A chainable continuum not homeomorphic to an inverse limit on* [0,1] *with only one bonding map*, Proc. Amer. Math. Soc. **18** (1967), 284–286.

5. S. Mardešić and J. Segal, *ε-mappings and inverse limits*, Trans. Amer. Math. Soc. **109** (1963), 146–164.

Emory University