ON THE ENTROPY OF CERTAIN CLASSES OF
SKEW-PRODUCT TRANSFORMATIONS

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1. Introduction. For the theory of Lebesgue spaces, skew-product
transformations and entropy see [5], [4] and [6]. Let X and Y be
Lebesgue spaces with measures μ and ν and let \( M = X \times Y \) be the
direct product space with the direct product measure \( \mu \times \nu \). Let \( S \) be
an automorphism of the space \( X \) and let \( \{ T_x \} \) be a family of auto-
morphisms of the space \( Y \), define \( U(x, y) = (Sx, T_x y) \). Under certain
measurability conditions on the family \( \{ T_x \} \) \( U \) is an automorphism
of the space \( M \). \( U \) is called the skew-product with base \( S \) and fibre
\( T_x \). The entropy of such a skew-product has been calculated [1], [4],
but the formulae given are not in general amenable to calculation.

We will consider two special types of skew-product:

1. Let \( \psi \) be an automorphism of \( X \) and let \( k(x) \) be an integer
valued function on \( X \) such that \( k(x) > 0 \) and \( \int_X k(x) \, d\mu < \infty \). The first
class of skew-products are those for which \( T_x = \psi^{k(x)} \).

2. Let \( \psi_t \) be a measurable measure-preserving flow on \( Y \) and let
\( f(x) \) be a real-valued function on \( X \) such that \( f(x) \geq r > 0 \), for some
constant \( r \), and \( \int_X f(x) \, d\mu < \infty \). The second class of skew-products are
those of the form \( T_x = \psi_f(x) \).

In both these special cases we will show that \( h(U) = h(S) + \int_X h(T_x) \, d\mu \). We note that the first class is not contained in the
second since the automorphism \( T \) may not be embeddable in a flow.

2. Induced automorphisms and special flows. Let \( T \) be an auto-
morphism of a Lebesgue space \( M \) with measure \( \mu \). If \( X \subset M \) and
\( \bigcup_{n=0}^{\infty} T^n X = M \), then \( T \) induces in the Lebesgue space \( X \), with mea-
sure \( \mu_X(A) = \mu(A) (\mu(X))^{-1} \), \( A \subset X \), the induced automorphism \( T' \)
given by \( T'x = T^{k(x)} x \) (\( x \in X \)) where \( k(x) \) is the smallest positive inte-
ger \( l \) such that \( T^l x \in X \). Abramov [2] has shown that \( h(T) = h(T') \mu(X) \).

Let us now consider the special flow built on an automorphism \( S \)
of a Lebesgue space \( M \) under a function \( f \). \( f \) is a real-valued function
on \( M \) such that

(i) \( f(x) \geq r > 0 \) for \( x \in M \),

(ii) \( \int_M f(x) \, d\mu < \infty \).

Let \( V \) be the subset of \( M \times R \), \( R \) is real line, defined by \( (x, u) \in V \)
if \( 0 \leq u < f(x) \). We define a group of automorphisms on \( V \) as follows;
for \( t < r \)

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\[ S_t(x, u) = (x, u + t) \quad \text{if } t < f(x) - u, \]

\[ = (Sx, t + u - f(x)) \quad \text{if } t \geq f(x) - u, \]

for the remaining values of \( t \) the automorphisms \( \{ S_t \} \) can be defined so that \( \{ S_t \} \) is a group. For an explicit formula for \( S_t \) see [6]. Abramov [3] has shown that \( h(S_t) = \frac{1}{\mu} h(S) \left( \int_M f(x) d\mu \right)^{-1} \). We note that the entropy of an automorphism is always calculated with respect to a normalised measure space.

3. Main theorems.

**Theorem 1.** Let \( U \) on \( X \times Y \) with measure \( \mu \times \nu \) be defined by \( U(x, y) = (Sx, T^{k(x)}y) \) where \( \infty > k(x) > 0 \), \( k(x) \) is integer valued and \( \int_X k(x) d\mu < \infty \). Then

\[ h(U) = h(S) + h(T) \int_X k(x) d\mu = h(S) + \int_X h(T^{k(x)}) d\mu. \]

**Proof.** Let \( Z \) denote the set of positive integers \( Z = \{1, 2, \ldots \} \) and let \( \eta \) be the measure on \( Z \) which assigns measure 1 to each point. Let \( V \) be the subset of \( X \times Y \times Z \) defined by \( (x, y, i) \in V \) if \( i \leq k(x) \). Note that \( V = V' \times Y \) where \( V' \) is the subset of \( X \times Z \) defined by \( (x, i) \in V' \) if \( i \leq k(x) \). It is easily seen that

\[ \mu \times \nu \times \eta(V) = \mu \times \eta(V') = \int_X k(x) d\mu. \]

Let us consider \( V \) as a Lebesgue space with the measure normalised, i.e., if \( A \subseteq V \) then

\[ \mu'(A) = \mu \times \nu \times \eta(A) \cdot \left( \int_X k(x) d\mu \right)^{-1}. \]

Define an automorphism \( \phi \) on \( V \) by

\[ \phi(x, y, i) = (x, Ty, i + 1) \quad \text{if } i < k(x), \]

\[ = (Sx, Ty, 1) \quad \text{if } i = k(x). \]

Then \( \phi = T \times \psi \) where \( \psi \) is the automorphism on \( V' \) defined by

\[ \psi(x, i) = (x, i + 1) \quad \text{if } i < k(x), \]

\[ = (Sx, 1) \quad \text{if } i = k(x). \]

It is clear that \( U \) is the automorphism induced by \( \phi \) on the set \( X \times Y \times \{1\} \). Thus we have

\[ h(U) = h(\phi)(\mu'(X \times Y \times \{1\}))^{-1} = \int_X k(x) d\mu h(\phi). \]
Since \( \phi = T \times \psi \) it follows that \( h(\phi) = h(T) + h(\psi) \) and it remains only to calculate \( h(\psi) \). It is clear that \( S \) is the automorphism induced by \( \psi \) on the set \( X \times \{1\} \); thus \( h(\psi) = h(S) \cdot (\int_X k(x) \, d\mu)^{-1} \). Finally

\[
h(U) = h(S) + \int_X k(x) \, d\mu \cdot h(T)
= h(S) + \int_X h(T^{k(x)}) \, d\mu.
\]

**Theorem 2.** Let \( U \) on \( X \times Y \) with measure \( \mu \times \nu \) be defined by \( U(x, y) = (Sx, \psi_f(y)) \) where \( f(x) \) is a real-valued function on \( X \), \( \infty > f(x) \geq \tau > 0 \) such that \( \int X f(x) \, d\mu < \infty \). Then

\[
h(U) = h(S) + \int_X f(x) \, d\mu \cdot h(\psi) = h(S) + \int_X h(\psi_{f(x)}) \, d\mu.
\]

**Proof.** Let \( R \) be the set of real numbers with the usual Lebesgue measure. Let \( A \) be the subset of \( X \times Y \times R \) defined by \((x, y, u) \in A \) if \( 0 \leq u < f(x) \) we note that \( A = A' \times Y \) where \( A' \) is the subset of \( X \times R \) defined by \((x, u) \in A' \) if \( 0 \leq u < f(x) \). We consider the following flows defined on \( A \), \( A' \) and \( A' \) respectively, all are defined for \( t < \tau \) and extended by group property,

\[
\phi_t(x, y, u) = (x, \psi_y, u + t) \quad \text{if } t < f(x) - u,
= (Sx, \psi_y, u + t - f(x)) \quad \text{if } t \geq f(x) - u;
\]

\[
\phi_t'(x, y, u) = (x, y, u + t) \quad \text{if } t < f(x) - u,
= (Sx, \psi_{f(y)}y, u + t - f(x)) \quad \text{if } t \geq f(x) - u;
\]

\[
\phi_t''(x, u) = (x, u + t) \quad \text{if } t < f(x) - u,
= (Sx, u + t - f(x)) \quad \text{if } t \geq f(x) - u.
\]

It is easy to see that \( \phi_t \) is a direct product of the flow \( \psi \), on \( Y \) with the flow \( \phi_t'' \) on \( A' \). It is also clear that \( \phi_t' \) and \( \phi_t'' \) are in the form of special flows built over the automorphisms \( U \) and \( S \) respectively and with the same function \( f(x) \), regarded firstly as a function of two variables \( f(x, y) = f(x) \) and secondly just as a function of one variable. Using Abramov's formula

\[
h(\phi_t') = \left| t \right| h(U) \left( \int_X f(x) \, d\mu \right)^{-1},
\]

\[
h(\phi_t'') = \left| t \right| h(S) \left( \int_X f(x) \, d\mu \right)^{-1}.
\]

Also \( h(\phi_t) = h(\psi_t) + h(\phi_t'') \). We will exhibit an isomorphism \( U \) of \( A \)
to itself which has the property that $U^{-1}\phi_1 U = \phi'_1$ for all $t$. Define

$$U(x, y, u) = (x, \psi y, u);$$

then, it is easily checked that $U$ satisfies the required property. Thus $h(\phi_1) = h(\phi'_1)$. Gathering together the formulae we have derived we obtain

$$h(\phi_1) = |t| h(U) \left( \int_x f(x) d\mu \right)^{-1} = h(\psi_1) + |t| h(S) \left( \int_x f(x) d\mu \right)^{-1}$$

or, in other words,

$$h(U) = h(S) + h(\psi_1) |t|^{-1} \int_x f(x) d\mu$$

$$= h(S) + h(\psi_1) \int_x f(x) d\mu,$$

since $h(\psi_1) = |t| h(\psi_1)$.

The proof of Theorem 1 depends on the restriction that $0 < k(x)$ and a natural question is to ask if the theorem is true when we let $k(x)$ take both positive and negative values. The following partial result was pointed out to us by W. Parry.

**Theorem 3.** Let $h(x)$ be an integer valued function on $X$ with the following properties

(i) $\int_X h(x) d\mu < \infty$.

(ii) There exists an integer-valued function $k(x)$ and an integrable integer-valued function $l(x)$ such that $h(x) = k(x) - l(x) + l(Sx)$ and $0 < k(x) < \infty$.

Then, if $U(x, y) = (Sx, T^{k(x)}y)$, $h(U) = h(S) + h(T) \int_X h(x) d\mu$.

We remark that it is possible to find functions $h(x)$ which take both positive and negative values and in such cases

$$h(T) \int_X h(x) d\mu \neq \int_X h(T^{k(x)}) d\mu.$$

**Proof.** Let $\phi(x, y) = (x, T^{l(x)}y)$; then

$$\phi^{-1}U\phi(x, y) = (Sx, T^{-l(Sx)}T^{k(x)}y)$$

$$= (Sx, T^{k(x)}y).$$

By Theorem 1,

$$h(U) = h(\phi^{-1}U\phi) = h(S) + h(T) \int_X k(x) d\mu.$$
But \( h(x) = k(x) - l(x) + l(Sx) \); therefore, since \( S \) preserves the measure \( \mu \),

\[
\int_X h(x) d\mu = \int_X k(x) d\mu.
\]

Thus \( h(U) = h(S) + h(T) \int_X h(x) d\mu \). A theorem of a similar type will also extend Theorem 2.

REFERENCES


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