ON THE ENTROPY OF CERTAIN CLASSES OF
SKEW-PRODUCT TRANSFORMATIONS

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1. Introduction. For the theory of Lebesgue spaces, skew-product transformations and entropy see [5], [4] and [6]. Let $X$ and $Y$ be Lebesgue spaces with measures $\mu$ and $\nu$ and let $M = X \times Y$ be the direct product space with the direct product measure $\mu \times \nu$. Let $S$ be an automorphism of the space $X$ and let $\{T_x\}$ be a family of automorphisms of the space $Y$, define $U(x, y) = (Sx, T_xy)$. Under certain measurability conditions on the family $\{T_x\}$ $U$ is an automorphism of the space $M$. $U$ is called the skew-product with base $S$ and fibre $T_x$. The entropy of such a skew-product has been calculated [1], [4], but the formulae given are not in general amenable to calculation.

We will consider two special types of skew-product:

1. Let $S$ be an automorphism of $X$ and let $k(x)$ be an integer valued function on $X$ such that $k(x) > 0$ and $\int_X k(x) d\mu < \infty$. The first class of skew-products are those for which $T_x = S^{k(x)}$.

2. Let $\psi_t$ be a measurable measure-preserving flow on $Y$ and let $f(x)$ be a real-valued function on $X$ such that $f(x) \geq \tau > 0$, for some constant $\tau$, and $\int_X f(x) d\mu < \infty$. The second class of skew-products are those of the form $T_x = \psi_{f(x)}$.

In both these special cases we will show that $h(U) = h(S) + \int_X h(T_x) d\mu$. We note that the first class is not contained in the second since the automorphism $T$ may not be embeddable in a flow.

2. Induced automorphisms and special flows. Let $T$ be an automorphism of a Lebesgue space $M$ with measure $\mu$. If $X \subseteq M$ and $\bigcup_{n=0}^{\infty} T^n X = M$, then $T$ induces in the Lebesgue space $X$, with measure $\mu_X(A) = \mu(A)(\mu(X))^{-1}$, $A \subseteq X$, the induced automorphism $T'$ given by $T'x = T^{k(x)}x$ ($x \in X$) where $k(x)$ is the smallest positive integer $l$ such that $T'^lx \in X$. Abramov [2] has shown that $h(T) = h(T')\mu(X)$.

Let us now consider the special flow built on an automorphism $S$ of a Lebesgue space $M$ under a function $f$. $f$ is a real-valued function on $M$ such that

(i) $f(x) \geq \tau > 0$ for $x \in M$,

(ii) $\int_M f(x) d\mu < \infty$.

Let $V$ be the subset of $M \times R$, $R$ is real line, defined by $(x, u) \in V$ if $0 \leq u < f(x)$. We define a group of automorphisms on $V$ as follows; for $t < \tau$

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\[ S_t(x, u) = (x, u + t) \text{ if } t < f(x) - u, \]
\[ = (Sx, t + u - f(x)) \text{ if } t \geq f(x) - u, \]

for the remaining values of \( t \) the automorphisms \( \{ S_t \} \) can be defined so that \( \{ S_t \} \) is a group. For an explicit formula for \( S_t \) see [6]. Abramov [3] has shown that \( h(S_t) = \int h(S) f(x) d\mu \). We note that the entropy of an automorphism is always calculated with respect to a normalised measure space.

3. Main theorems.

**Theorem 1.** Let \( U \) on \( X \times Y \) with measure \( \mu \times \nu \) be defined by \( U(x, y) = (Sx, T^{k(x)}y) \) where \( k(x) > 0 \), \( k(x) \) is integer valued and \( \int x k(x) d\mu < \infty \). Then

\[ h(U) = h(S) + h(T) \int x k(x) d\mu = h(S) + \int x h(T^{k(x)}) d\mu. \]

**Proof.** Let \( Z \) denote the set of positive integers \( Z = \{ 1, 2, \ldots \} \) and let \( \eta \) be the measure on \( Z \) which assigns measure 1 to each point. Let \( V \) be the subset of \( X \times Y \times Z \) defined by \((x, y, i) \in V \) if \( i \leq k(x) \). Note that \( V = V' \times Y \) where \( V' \) is the subset of \( X \times Z \) defined by \((x, i) \in V' \) if \( i \leq k(x) \). It is easily seen that

\[ \mu \times \nu \times \eta(V) = \mu \times \eta(V') = \int x k(x) d\mu. \]

Let us consider \( V \) as a Lebesgue space with the measure normalised, i.e., if \( A \subseteq V \) then

\[ \mu'(A) = \mu \times \nu \times \eta(A) \cdot \left( \int x k(x) d\mu \right)^{-1}. \]

Define an automorphism \( \phi \) on \( V \) by

\[ \phi(x, y, i) = (x, Ty, i + 1) \text{ if } i < k(x), \]
\[ = (Sx, Ty, 1) \text{ if } i = k(x). \]

Then \( \phi = T \times \psi \) where \( \psi \) is the automorphism on \( V' \) defined by

\[ \psi(x, i) = (x, i + 1) \text{ if } i < k(x), \]
\[ = (Sx, 1) \text{ if } i = k(x). \]

It is clear that \( U \) is the automorphism induced by \( \phi \) on the set \( X \times Y \times \{ 1 \} \). Thus we have

\[ h(U) = h(\phi)(\mu'(X \times Y \times \{ 1 \}))^{-1} = \int x k(x) d\mu h(\phi). \]
Since $\phi = T \times \psi$ it follows that $h(\phi) = h(T) + h(\psi)$ and it remains only to calculate $h(\psi)$. It is clear that $S$ is the automorphism induced by $\psi$ on the set $X \times \{1\}$; thus $h(\psi) = h(S) \cdot (\int_X k(x) \, d\mu)^{-1}$. Finally

$$h(U) = h(S) + \int_X h(x) \, d\mu \cdot h(T)$$

$$= h(S) + \int_X h(T(x)) \, d\mu.$$  

**Theorem 2.** Let $U$ on $X \times Y$ with measure $\mu \times \nu$ be defined by $U(x, y) = (Sx, \psi_f(x)y)$ where $f(x)$ is a real-valued function on $X$, $\infty > f(x) \geq \tau > 0$ such that $\int_X f(x) \, d\mu < \infty$. Then

$$h(U) = h(S) + \int_X f(x) \, d\mu h(\psi_f) = h(S) + \int_X h(\psi_f(x)) \, d\mu.$$  

**Proof.** Let $R$ be the set of real numbers with the usual Lebesgue measure. Let $A$ be the subset of $X \times Y \times R$ defined by $(x, y, u) \in A$ if $0 \leq u < f(x)$ we note that $A = A' \times Y$ where $A'$ is the subset of $X \times R$ defined by $(x, u) \in A'$ if $0 \leq u < f(x)$. We consider the following flows defined on $A$, $A$ and $A'$ respectively, all are defined for $t < \tau$ and extended by group property,

$$\phi_t(x, y, u) = (x, \psi_t y, u + t) \quad \text{if } t < f(x) - u,$$

$$= (Sx, \psi_t y, u + t - f(x)) \quad \text{if } t \geq f(x) - u;$$

$$\phi'_t(x, y, u) = (x, y, u + t) \quad \text{if } t < f(x) - u,$$

$$= (Sx, \psi'_t(x)y, u + t - f(x)) \quad \text{if } t \geq f(x) - u;$$

$$\phi''_t(x, u) = (x, u + t) \quad \text{if } t < f(x) - u,$$

$$= (Sx, u + t - f(x)) \quad \text{if } t \geq f(x) - u.$$  

It is easy to see that $\phi_t$ is a direct product of the flow $\psi_t$ on $Y$ with the flow $\phi''_t$ on $A'$. It is also clear that $\phi'_t$ and $\phi''_t$ are in the form of special flows built over the automorphisms $U$ and $S$ respectively and with the same function $f(x)$, regarded firstly as a function of two variables $f(x, y) = f(x)$ and secondly just as a function of one variable. Using Abramov's formula

$$h(\phi_t) = |t| h(U) \left( \int_X f(x) \, d\mu \right)^{-1},$$

$$h(\phi'_t) = |t| h(S) \left( \int_X f(x) \, d\mu \right)^{-1}.$$  

Also $h(\phi_t) = h(\psi_t) + h(\phi''_t)$. We will exhibit an isomorphism $U$ of $A$
to itself which has the property that $U^{-1}\phi_t U = \phi_t'$ for all $t$. Define
\[ U(x, y, u) = (x, \psi_uy, u); \]
then, it is easily checked that $U$ satisfies the required property. Thus
\[ h(\phi_t) = h(\phi_t'). \]
Gathering together the formulae we have derived we obtain
\[ h(\phi_t) = \left| t \right| h(U) \left( \int_x f(x)d\mu \right)^{-1} = h(\psi_t) + \left| t \right| h(S) \left( \int_x f(x)d\mu \right)^{-1} \]
or, in other words,
\[ h(U) = h(S) + h(\psi_t) \left| t \right|^{-1} \int_x f(x)d\mu \]
\[ = h(S) + h(\psi_t) \int_x f(x)d\mu, \]
since $h(\psi_t) = |t| h(\psi_t)$.

The proof of Theorem 1 depends on the restriction that $0 < k(x)$ and a natural question is to ask if the theorem is true when we let $k(x)$ take both positive and negative values. The following partial result was pointed out to us by W. Parry.

**Theorem 3.** Let $h(x)$ be an integer valued function on $X$ with the following properties
i) \( \int_X h(x)d\mu < \infty. \)
ii) There exists an integer-valued function $k(x)$ and an integrable integer-valued function $l(x)$ such that $h(x) = k(x) - l(x) + l(Sx)$ and $0 < k(x) < \infty$.

Then, if $U(x, y) = (Sx, T^{k(x)}y)$, $h(U) = h(S) + h(T) \int x h(x)d\mu$.

We remark that it is possible to find functions $h(x)$ which take both positive and negative values and in such cases
\[ h(T) \int x h(x)d\mu \neq \int x h(T^{k(x)})d\mu. \]

**Proof.** Let $\phi(x, y) = (x, T^{l(x)}y)$; then
\[ \phi^{-1}U\phi(x, y) = (Sx, T^{-1}(Sx) T^{k(x)} T^{l(x)}y) \]
\[ = (Sx, T^{k(x)}y). \]

By Theorem 1,
\[ h(U) = h(\phi^{-1}U\phi) = h(S) + h(T) \int x k(x)d\mu. \]
But $h(x) = k(x) - l(x) + l(Sx)$; therefore, since $S$ preserves the measure $\mu$,

$$\int_x h(x) d\mu = \int_x k(x) d\mu.$$ 

Thus $h(U) = h(S) + h(T) \int_x h(x) d\mu$. A theorem of a similar type will also extend Theorem 2.

**References**


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